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Classification of type III Bernoulli crossed products

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Dissertation presented in partial fulfillment of the requirements for the degree of Doctor of Science (PhD): Mathematics

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Abstract

Crossed products with noncommutative Bernoulli actions were introduced by Connes as the first examples of full factors of type III. In this thesis, partially published in [VV14, Ver15], we provide a complete classification of the factors $(P, \phi)^{\mathbb{F}_n} \rtimes \mathbb{F}_n$, where $\mathbb{F}_n$ is the free group and P is an amenable factor with a normal faithful state ϕ that either is almost periodic, or has a weakly mixing modular automorphism group. We show that the family of factors $(P, \phi)^{\mathbb{F}_n} \rtimes \mathbb{F}_n$ with ϕ almost periodic, is completely classified by the rank n of the free group $\mathbb{F}_n$ and Connes's Sd-invariant; and that the family of factors $(P, \phi)^{\mathbb{F}_n} \rtimes \mathbb{F}_n$ with ϕ a weakly mixing state, is classified by n and the action $\mathbb{F}_n \curvearrowright (P, \phi)^{\mathbb{F}_n}$, up to state-preserving conjugation of the action. We prove similar results for free product groups, as well as for classes of generalized Bernoulli actions.

Beknopte samenvatting

Gekruiste producten van niet-commutatieve Bernoulli-acties werden door Connes geïntroduceerd als de eerste voorbeelden van volle factoren van type III. In dit proefschrift, dat deels gepubliceerd werd in [VV14, Ver15], geven we een volledige classificatie van de factoren $(P,\phi)^{\mathbb{F}_n} \rtimes \mathbb{F}_n$, waarbij $\mathbb{F}_n$ een vrije groep is en P een amenabele factor, uitgerust met een normale trouwe toestand ϕ die ofwel bijna periodisch is, ofwel een *weakly mixing* modulaire automorfismegroep heeft. We tonen aan dat de familie van factoren $(P,\phi)^{\mathbb{F}_n} \rtimes \mathbb{F}_n$ met ϕ een bijna periodische toestand, volledig geclassificeerd wordt door de rang n van de vrije groep en door Connes' Sd-invariant. Verder bewijzen we dat de familie van factoren $(P,\phi)^{\mathbb{F}_n} \rtimes \mathbb{F}_n$ waarbij ϕ *weakly mixing* is, geclassificeerd wordt door n en de actie $\mathbb{F}_n \curvearrowright (P,\phi)^{\mathbb{F}_n}$, op een toestandbewarende conjugatie van de actie na. We vermelden verder gelijkaardige resultaten voor vrije productgroepen en voor klasses van veralgemeende Bernoulli-acties.

Contents

Chapter 1

Introduction

In this chapter, we give a brief introduction to von Neumann algebras, and present our main results. This thesis is based on our joint work with Stefaan Vaes [VV14] and on our single-author paper [Ver15]. In particular, parts of this chapter originate from the introductions of these articles.

1.1 Von Neumann algebras

A *von Neumann algebra* is a unital $\star$-subalgebra of $B(H)$ for some Hilbert space H that is closed for the *σ-weak topology*, which is the weak-$\star$ topology obtained from the predual $B_\star(H)$ of $B(H)$, i.e. the weakest topology such that all trace class operators S are continuous when considered as functions $\mathrm{Tr}(S\,\cdot)$ on $B(H)$. Von Neumann algebras can also be algebraically described, as follows. For any subset $N \subset B(H)$, we denote by N' the *commutant* of N inside $B(H)$, given by $N' = \{S \in B(H) \mid ST = TS \text{ for all } T \in N\}$. By von Neumann's famous bicommutant theorem [vN29], a unital $\star$-subalgebra $M \subset B(H)$ is closed for the σ-weak operator topology, if and only if $M = (M')'$. As a corollary, von Neumann algebras are also closed for the (stronger) weak and strong operator topologies. Basic examples of von Neumann algebras are matrix algebras $M_n(\mathbb{C})$ or abelian von Neumann algebras $L^\infty(X,\mu)$, consisting of bounded measurable functions from a measure space (X,μ) to $\mathbb{C}$.

We say that two von Neumann algebras M and N are isomorphic, if there exists a $\star$-isomorphism $M \to N$. Such a $\star$-isomorphism is automaticly continuous [Ped89, Exercise E.4.6.10] for the σ-weak topologies on M and N, which indicates that the σ-weak topology is the 'natural topology' to consider on von Neumann algebras. Conversely, from Kaplansky's density theorem it follows that if a $\star$-homomorphism $\pi : M \to B(H)$ of a von Neumann algebra into some $B(H)$ is σ-weakly continuous, then $\pi(M) \subset B(H)$ is also a von Neumann algebra.

A *factor* is a von Neumann algebra M with trivial center $\mathcal{Z}(M) = M \cap M'$. Factors serve as the basic building blocks of von Neumann algebras: by means of a direct integral decomposition, any von Neumann algebra can be written as a generalized 'direct sum' of factors.

We now introduce two ways to build new von Neumann algebras from old ones.

Tensor products If M, N are von Neumann algebras represented on Hilbert spaces H, K respectively, then the *tensor product* of M and N is the von Neumann algebra on $H \otimes K$ given by

$$M \mathbin{\overline{\otimes}} N = \{x \otimes y \mid x \in M, y \in N\}''.$$

One can also form the tensor product of a countable infinite family of von Neumann algebras $(P_i)_{i \in I}$, provided that one fixes a normal faithful state on each von Neumann algebra P_i, see Definition 2.24 below. We now only note that the choice of these states is required to give a useful definition, and that it highly influences the structure of the resulting von Neumann algebra.

Crossed products Whenever Γ is a countable discrete group that acts on a von Neumann algebra M by $\star$-automorphisms, i.e. whenever we have a homomorphism $\Gamma \to \mathrm{Aut}(M) : g \mapsto \alpha_g$ to the group of $\star$-isomorphisms $M \to M$, one can form a new von Neumann algebra $M \rtimes \Gamma$ encoding this action as follows: $M \rtimes \Gamma$ is the unique von Neumann algebra generated by a copy of M and by unitaries u_g, $g \in \Gamma$, such that $u_{gh} = u_g u_h$ and $u_g^\star = u_{g^{-1}}$ for all $g, h \in \Gamma$, such that

$$u_g x = \alpha_g(x) u_g, \qquad \text{for all } x \in M, g \in \Gamma,$$

and such that there exists a conditional expectation $E : M \rtimes \Gamma \to M$ satisfying $E(u_g) = 0$ for $g \in \Gamma, g \neq e$. The resulting von Neumann algebra $M \rtimes \Gamma$ is called the *crossed product*. For a construction, as well as for the introduction of crossed products for actions of locally compact groups, we refer to Section 2.1.2.

By studying their projections, von Neumann algebras can be classified into three types: type I, II and III. A factor is of type I if it is of the form $B(H)$ for a Hilbert space H, of type II if it is not of the form $B(H)$ but has a tracial semifinite weight, and of type III otherwise. See Section 2.1 for a more detailed description of this type classification.

Before we continue, we state the following common assumption.

Assumption. In this thesis, all mentioned von Neumann algebras are assumed to have separable predual; or equivalently, they are assumed to have a faithful representation on a separable Hilbert space.

1.2 Isomorphism and non-isomorphism results

Classifying von Neumann algebras up to isomorphism has always been a central theme in von Neumann algebra theory. While Murray and von Neumann's first papers [MVN35, MvN36] focussed on the type classification of von Neumann algebras — at that time called *rings of operators* — into the three classes I, II and III, and on providing examples of factors of type II [MVN35] and type III [vN39], the classification problem already turns up in the fourth paper on the subject in 1943. In [MvN43], they show that there is a unique *hyperfinite* II_1 factor, i.e. that all factors of type II_1 that can be written as the strong (or equivalently, σ-weak) closure of an increasing union of finite-dimensional subalgebras, are isomorphic. This unique factor can explicitly be constructed as the infinite tensor product of two-by-two matrices:

$$R = \overline{\bigotimes}_{n\in\mathbb{N}}(M_2(\mathbb{C}), \mathrm{tr}),$$

where tr is the normalized trace on $M_2(\mathbb{C})$. In the same paper, they are also able to distinguish $R \ncong L(\mathbb{F}_2)$ as two non-isomorphic II_1 factors, by introducting the so-called *property* Γ.

The study of hyperfinite II_1 factors inspired Powers to consider hyperfinite factors of type III as well, and he showed [Pow67] that the family

$$R_\lambda = \overline{\bigotimes}_{n\in\mathbb{N}}(M_2(\mathbb{C}), \varphi_\lambda), \quad \text{with } \varphi_\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{1+\lambda}a + \frac{\lambda}{1+\lambda}d,$$

for $\lambda \in (0,1)$ yields pairwise non-isomorphic factors of type III. His discovery was historically followed by the development of modular theory by Tomita [Tom67] in 1967, but the power of this theory was only understood when Takesaki published a refined version in [Tak70]. As a consequence of this modular theory, type III factors could be distinguished in subtypes III_λ, for $\lambda \in [0,1]$, and the discovery of Powers was now reduced to a direct consequence of the type III subclassification: Powers's factor R_λ is of type III_λ. Further refinements of the modular theory eventually lead to Connes's Sd and τ-invariants [Con74], allowing him to give examples — the noncommutative Bernoulli crossed products — of non-isomorphic type III_1 factors.

Up to this point in history, the focus has been on distinguishing von Neumann algebras by exploiting *modular properties*, i.e. the type classification and the existence of certain types of weights, but the classification of factors inside the same class, in particular the classification of type II_1, remained wide open. Only 30 years after the first discovery of the two non-isomorphic II_1 factors $L(S_\infty)$ and $L(\mathbb{F}_2)$, McDuff [McD69] showed the existence of an uncountable family of pairwise non-isomorphic II_1 factors (see also Sakai [Sak69] for a similar result). A major advance in the study of type II_1 was Connes's groundbreaking result that hyperfiniteness is equivalent to *injectivity* of a von Neumann algebra [Con75]. A von Neumann algebra $M \subset B(H)$ is *injective*, or *amenable*, if there exists a conditional expectation $B(H) \to M$ (see Page 19 for the definition). As a result, Connes could show that there exists a unique injective factor of type II_1, type II_∞ and type III_λ with $\lambda \in (0,1)$, and in particular, that all injective III_λ factors are isomorphic to Powers's factor R_λ, for $\lambda \in (0,1)$. Combined with earlier work by Krieger [Kri75], he also showed that the uncountably many injective type III_0 factors can be classified by *ergodic flows*. The classification problem for injective III_1 factors was left open, and was settled a decade later by Haagerup [Haa85], who showed the uniqueness of the injective factor of type III_1 by studying bicentralizers.

As beautiful as the uniqueness result for injective factors is, it also indicates the difficulty of finding non-isomorphic factors of the same type, and of determining the border between 'isomorphism' and 'non-isomorphism' phenomena. In that respect, Popa's deformation/rigidity theory [Pop01, Pop03, Pop04], developed in the early 2000's, is a major breaktrough, as it provides a basis for several structural results, classifying certain families of von Neumann algebras in terms of their building data, such as group measure space constructions $L^\infty(X) \rtimes \Gamma$ associated with probability measure preserving (pmp) free ergodic actions. For example, in [PV11], it was proved that the crossed product $L^\infty(X) \rtimes \mathbb{F}_n$ with an *arbitrary* free ergodic pmp action of the free group $\mathbb{F}_n$ has $L^\infty(X)$ as its unique Cartan subalgebra up to unitary conjugacy. In combination with work of Gaboriau [Gab01] and Bowen [Bow09], this yielded the classification of the Bernoulli crossed products $L^\infty(X_0^{\mathbb{F}_n}) \rtimes \mathbb{F}_n$ of the free groups, showing that the rank of $\mathbb{F}_n$ is a complete invariant. In particular, the specific choice of the base probability space (X_0, μ_0) plays no role.

As we saw above, classifying a family of von Neumann algebras involves two steps: distinguishing between the non-isomorphic ones and proving that the others are isomorphic. For the first step, a lot of modern results were obtained in the framework of Popa's deformation/rigidity theory, and in the type III case, these must be combined with the modular theory of Connes and Takesaki and the corresponding invariants for type III factors. The main source for the second step is of a different nature and ultimately makes use of Connes's discovery that in the amenable case, “everything is isomorphic” : amenable factors are completely classified by their type [Con75, Haa85], free ergodic pmp actions of amenable groups are orbit equivalent [OW80], and outer actions of amenable groups on the hyperfinite II_1 factor are cocycle conjugate [Ocn85]. Only in rare circumstances, both steps can be fully achieved simultaneously. In our work, we will study the classification of noncommutative Bernoulli crossed products, a very relevant family of von Neumann algebras, as it yielded important examples of non-isomorphic type III_1 factors. The classification of these factors requires a combination of ‘old’ modular theory with the modern deformation/rigidity theory.

1.3 Overview of our main results

Bernoulli crossed products were introduced by Connes [Con74] as the first examples of full factors of type III; we briefly present their construction. Let (P, ϕ) be any von Neumann algebra equipped with a normal faithful state ϕ, and let Λ be a countably infinite group. Then take the infinite tensor product $P^\Lambda = \overline{\bigotimes}_{g\in\Lambda} P$ indexed by Λ, with respect to the state ϕ. The group Λ acts on P^Λ by shifting the tensor factors. This action is called a *noncommutative Bernoulli action*. The factor $P^\Lambda \rtimes \Lambda$ is called a *Bernoulli crossed product*. If ϕ is not a trace, $P^\Lambda \rtimes \Lambda$ is a type III factor. We always assume that the base algebra P is an amenable factor.

A first and obvious invariant for the Bernoulli crossed products $P^\Lambda \rtimes \Lambda$, is amenability. If Λ is infinite amenable, then by [Con75, Haa85], the Bernoulli crossed product $P^\Lambda \rtimes \Lambda$ is completely determined by its type : $P^\Lambda \rtimes \Lambda$ is of type II_1 if ϕ is a trace, of type III_λ for $\lambda \in (0, 1)$ if ϕ is periodic with period $\frac{2\pi}{|\log \lambda|}$, and of type III_1 if ϕ is a nonperiodic state (the crossed product $P^\Lambda \rtimes \Lambda$ is a factor by Corollary 2.31, and see Remark 2.32 and Lemma 4.3 for the type classification).

If Λ is nonamenable, the Bernoulli crossed product $P^\Lambda \rtimes \Lambda$ is a full factor, see [Con74] and Lemma 2.34. In particular, we can use Connes's Sd-invariants to further distinguish among the Bernoulli crossed products of type III_1, at least when the state on the base algebra is *almost periodic*. A state ϕ on a von Neumann algebra P is almost periodic if the modular operator Δ_ϕ on $L^2(P, \phi)$ is diagonalizable, and in this case, we denote by $\Gamma(P, \phi)$ the multiplicative subgroup of $\mathbb{R}_0^+$ generated by the point spectrum of the modular operator Δ_ϕ. If the state ϕ on P is almost periodic, then Connes's Sd-invariant of the crossed product $(P, \phi)^\Lambda \rtimes \Lambda$ exactly equals $\Gamma(P, \phi)$; otherwise, if ϕ is not almost periodic, then $\mathrm{Sd}(P^\Lambda \rtimes \Lambda) = \mathbb{R}_0^+$ (see Section 2.1.5 and Remark 4.4).

Using Popa's deformation/rigidity theory, and in particular the results of [PV12], we show that the group Λ is also an invariant among all the Bernoulli crossed products $P^\Lambda \rtimes \Lambda$ with Λ belonging to a large class of nonamenable, countable groups including all nonelementary hyperbolic groups (see Definition 2.18). Conversely, combining Ocneanu's classification of outer actions of amenable groups on amenable factors [Ocn85] and Bowen's co-induction argument [Bow09], we prove that the factors $(P_i, \phi_i)^\Lambda \rtimes \Lambda$, $i = 0, 1$, are isomorphic whenever $\Lambda = \Sigma \star \Upsilon$ is a free product with Σ infinite amenable, the states ϕ_i are almost periodic, and $\Gamma(P_0, \phi_0) = \Gamma(P_1, \phi_1)$.

Altogether, this gives our first main result.

Theorem A. *The set of factors*

$$\left\{(P,\phi)^\Lambda \rtimes \Lambda \;\middle|\; \begin{array}{l} P \text{ a nontrivial amenable factor with normal faithful almost} \\ \text{periodic state } \phi, \text{ and } \Lambda = \Sigma \star \Upsilon \text{ a free product of an infinite} \\ \text{amenable group } \Sigma \text{ and a nontrivial countable group } \Upsilon \end{array}\right\}$$

is exactly classified, up to isomorphism, by $\Gamma(P,\phi) \subset \mathbb{R}_0^+$ and the isomorphism class of Λ.

Concretely, Theorem A implies that the Bernoulli crossed products of the free groups

$$\left\{(M_k(\mathbb{C}),\phi)^{\mathbb{F}_n} \rtimes \mathbb{F}_n \;\middle|\; \begin{array}{l} n \geq 2,\ k \geq 1, \text{ and } \phi(x) = \mathrm{Tr}(\mathrm{diag}(\mu_1, \cdots, \mu_k)x) \\ \text{with } \mu_i > 0 \text{ and } \sum_{i=1}^k \mu_i = 1 \end{array}\right\}$$

are exactly classified, up to isomorphism, by n and the subgroup of $\mathbb{R}_0^+$ generated by the ratios $\frac{\mu_i}{\mu_j}$, for $1 \leq i, j \leq k$.

The above classification of the noncommutative Bernoulli crossed products complements earlier classification results for other families of type III factors, such as group measure space constructions for nonsingular actions [HV12], Shlyakhtenko's free Araki-Woods factors [Shl96, Hou08] and free quantum group factors [Iso14]. However, obtaining a full classification for families of type III factors without using the existence of almost periodic states, is extremely difficult. Even the free Araki-Woods factors are only fully classified when the underlying one-parameter group is almost periodic [Shl96]. The next result overcomes these difficulties, and provides a non-isomorphism result for Bernoulli crossed products $(P,\phi)^\Lambda \rtimes \Lambda$ where the state ϕ is weakly mixing, i.e. the modular operator Δ_ϕ of the state ϕ has no nontrivial finite-dimensional invariant subspaces. The class $\mathcal{C}$ of countable groups is introduced in Definition 2.18 and by Remark 2.19, it contains all nonelementary hyperbolic groups, as well as all nontrivial free product groups.

Theorem B. *The set of factors*

$$\Big\{ (P,\phi)^\Lambda \rtimes \Lambda \;\Big|\; \begin{array}{l} P \text{ a nontrivial amenable factor with a normal faithful weakly} \\ \text{mixing state } \phi, \text{ and } \Lambda \text{ an icc group in the class } \mathcal{C} \end{array} \Big\}$$

is exactly classified, up to isomorphism, by the group Λ and the action $\Lambda \curvearrowright (P,\phi)^\Lambda$ up to a state-preserving conjugacy of the action.

Examples of normal faithful weakly mixing states ϕ on the hyperfinite III_1 factor arise from Araki–Woods representations of the canonical commutation relations (CCR) [Ara63] and from Araki–Wyss representations of the canonical anticommutation relations (CAR) [AW64]. We quickly discuss the latter. The *CAR*-functor associates to each self-adjoint operator $\rho \in B(H)$ with $0 \leq \rho \leq 1$ on a real Hilbert space H, an amenable factor $R(\rho)$ with a normal faithful state ϕ. The modular operator of this state ϕ can explicitly be described in terms of ρ, and in particular, the state ϕ on $R(\rho)$ is weakly mixing if and only if ρ has no point spectrum (see also Section 4.5). In this case, $R(\rho)$ is the hyperfinite III_1 factor. In Section 4.5 below, we show the existence of two self-adjoint operators ρ_1 and ρ_2 such that the Bernoulli crossed products $R(\rho_i)^\Lambda \rtimes \Lambda$ can be distinguished by Theorem B, but not by earlier invariants such as Connes's τ-invariant [Con74].

Both the 'non-isomorphism part' of Theorem A and Theorem B are particular cases of the following optimal classification result for general states ϕ on the base algebra. For every nontrivial factor (P,ϕ) equipped with a normal faithful state, we denote by $P_{\phi,\mathrm{ap}} \subset P$ the *almost periodic part* of P, i.e. $P_{\phi,\mathrm{ap}}$ is the subalgebra spanned by the eigenvectors of Δ_ϕ,

$$P_{\phi,\mathrm{ap}} = \Big(\operatorname{span} \bigcup_{\mu \in \mathbb{R}_0^+} \{x \in P \mid \sigma_t^\phi(x) = \mu^{\mathbf{i}t} x\}\Big)''.$$

We obtain the following main theorem, classifying all Bernoulli crossed products with amenable factors (P,ϕ) as base algebra, under the assumption that the almost periodic part of the base algebra, $P_{\phi,\mathrm{ap}}$, is a factor. This assumption is equivalent to the assumption that the centralizer $(P^I)_{\phi^I}$ of the infinite tensor product P^I with respect to ϕ^I is a factor, see Lemma 2.29 below.

Theorem C. *Let (P_0,ϕ_0) and (P_1,ϕ_1) be nontrivial amenable factors equipped with normal faithful states, such that $(P_0)_{\phi_0,ap}$ and $(P_1)_{\phi_1,ap}$ are factors. Let Λ_0 and Λ_1 be icc groups in the class $\mathcal{C}$.*

The algebras $P_0^{\Lambda_0} \rtimes \Lambda_0$ and $P_1^{\Lambda_1} \rtimes \Lambda_1$ are isomorphic if and only if one of the following statements holds.

(a) *The states ϕ_0 and ϕ_1 are both tracial, and the actions $\Lambda_i \curvearrowright (P_i,\phi_i)^{\Lambda_i}$ are cocycle conjugate, modulo a group isomorphism $\Lambda_0 \cong \Lambda_1$.*

(b) *The states ϕ_0 and ϕ_1 are both nontracial, and there exist projections $p_i \in (P_i^{\Lambda_i})_{\phi_i^{\Lambda_i}}$ such that the reduced cocycle actions $(\Lambda_i \curvearrowright (P_i, \phi_i)^{\Lambda_i})^{p_i}$ are cocycle conjugate through a state-preserving isomorphism, modulo a group isomorphism $\Lambda_0 \cong \Lambda_1$.*

Note that if the centralizers of $P_i^{\Lambda_i}$ with respect to $\phi_i^{\Lambda_i}$ are trivial, then we automatically get conjugation of the two actions. In particular, Theorem B is now a direct consequence of Theorem C, since if ϕ is a weakly mixing state on P, then $P_{\phi,\mathrm{ap}} = \mathbb{C}$ is the trivial factor, and hence the centralizer of $P_i^{\Lambda_i}$ with respect to $\phi_i^{\Lambda_i}$ is also trivial. More generally, if the group Λ_i is a direct product of two icc groups in the class $\mathcal{C}$, we can apply Popa's cocycle superrigidity theorems (see Appendix A) and also get conjugation. We obtain the following result.

Theorem D. *The set of factors*

$$\Big\{(P,\phi)^{\Lambda} \rtimes \Lambda \;\Big|\; \begin{array}{l} P \textit{ a nontrivial amenable factor with normal faithful state } \phi \\ \textit{such that } P_{\phi,ap} \textit{ is a factor, and } \Lambda \textit{ a direct product of two} \\ \textit{icc groups in the class } \mathcal{C} \end{array} \Big\}$$

is exactly classified, up to isomorphism, by the group Λ and the action $\Lambda \curvearrowright (P,\phi)^{\Lambda}$ up to a state-preserving conjugacy of the action.

In Theorems B and D, we have shown that certain families of Bernoulli crossed products are classified by the conjugacy class of the Bernoulli actions. Unfortunately, we do not understand in general when two Bernoulli actions $\Lambda \curvearrowright (P_0,\phi_0)^{\Lambda}$ and $\Lambda \curvearrowright (P_1,\phi_1)^{\Lambda}$ are conjugate through a state-preserving isomorphism. However, we believe that in the case when the states ϕ_i are almost periodic, the existence of a state-preserving conjugacy between the actions $\Lambda \curvearrowright (P_i,\phi_i)^{\Lambda}$ is a strictly stronger condition than the equality of the groups $\Gamma(P_i,\phi_i)$ that appears in Theorem A. This can be best illustrated when $\Lambda = \mathbb{Z}$ and (P,ϕ) is a factor of type I with a normal faithful state. According to the work of Connes and Størmer [CS74] and Connes, Narnhofer and Thirring [CNT87], apart from the subgroup $\Gamma(P,\phi) \subset \mathbb{R}_0^+$, also the entropy $H(\phi)$ is a state-preserving conjugacy invariant of $\Lambda \curvearrowright (P,\phi)^{\Lambda}$. Here the entropy of a normal faithful state ϕ on a type I factor P is given by

$$H(\phi) = -\sum_k \phi(p_k)\log\phi(p_k),$$

where the p_k form a maximal orthogonal family of minimal projections in the centralizer P_ϕ.

It is highly plausible that for large classes of infinite groups Λ and type I factors P, the entropy $H(\phi)$ is a state-preserving conjugacy invariant of $\Lambda \curvearrowright$

$(P, \phi)^\Lambda$. Some evidence for this can be found in the commutative case, as Bowen [Bow08a, Bow08b] showed that the entropy μ_0 of the base probability space (X_0, μ_0) is a conjugacy invariant for Bernoulli actions $\Lambda \curvearrowright (X_0, \mu_0)^\Lambda$ of sofic groups. Furthermore, for every countably infinite group Λ, two Bernoulli actions $\Lambda \curvearrowright X_0^\Lambda$ and $\Lambda \curvearrowright Y_0^\Lambda$ were shown to be conjugate once $(X, \mu_0), (Y, \nu_0)$ are both not two-atomic and $H(\mu_0) = H(\nu_0)$, see [Orn70a, Orn70b, Bow11].

One can thus even speculate that for large classes of countable groups Λ, the existence of a state-preserving conjugacy between two Bernoulli actions $\Lambda \curvearrowright (P_i, \phi_i)^\Lambda$, $i = 0, 1$, with P_i type I factors and ϕ_i normal faithful states, is equivalent with the equality of both $\Gamma(P_0, \phi_0) = \Gamma(P_1, \phi_1)$ and $H(\phi_0) = H(\phi_1)$. To prove such a statement, there is a two-fold open problem to be solved. Is Connes-Narnhofer-Thirring entropy a conjugacy invariant for noncommutative Bernoulli actions of, say, sofic groups? In other words, is it possible to develop a noncommutative version of Bowen's sofic entropy? Secondly, do equal entropy and equal point spectrum imply conjugacy of the actions? In other words, is there a noncommutative Ornstein theorem? This is a 'classical' problem that is open for the group $\mathbb{Z}$. But once it is solved for $\mathbb{Z}$, by co-induction, the same result follows for all groups containing $\mathbb{Z}$.

In contrast with the situation above, a conjugacy between *two-sided* Bernoulli actions $\Lambda \times \Lambda \curvearrowright (P_0, \phi_0)^\Lambda$ and $\Lambda \times \Lambda \curvearrowright (P_1, \phi_1)^\Lambda$ for an icc group Λ yields more information, as it provides an isomorphism between the base algebras. Thus, for two-sided Bernoulli actions, we get the following optimal result.

Theorem E. *The set of factors*

$$\Big\{(P, \phi)^\Lambda \rtimes (\Lambda \times \Lambda) \;\Big|\; \begin{array}{l} P \textit{ a nontrivial amenable factor with normal faithful} \\ \textit{state } \phi \textit{ such that } P_{\phi,ap} \textit{ is a factor, and } \Lambda \textit{ an icc} \\ \textit{group in the class } \mathcal{C} \end{array} \Big\}$$

is exactly classified, up to isomorphism, by the group Λ and the pair (P, ϕ) up to a state-preserving isomorphism.

We conclude this section with an outline of the following chapters. In the next chapter, Chapter 2, we introduce further notions we will need to prove our classification results, and we prove several structural properties that Bernoulli actions and their associated crossed products have. In Chapter 3, we discuss actions of groups on factors that preserve an almost periodic state, and we prove Theorem A and give a partial proof for Theorems D and E restricting to almost periodic states on the base algebra. For the latter, we rely on Popa's cocycle superrigidity theorem for noncommutative Bernoulli actions, which we will discuss below. In Chapter 4, we study Bernoulli crossed products for which the base algebra does not necessarily carry an almost periodic state, which allows us to show Theorems B and C, and to complete the proof of Theorems D and E for general states on the base algebra.

In the proofs of Theorems D and E, we need to go from cocycle conjugacy to conjugacy for actions of the form $\Lambda \curvearrowright (P, \phi)^I$. For this, we must prove that all 1-cocycles for the action are trivial. In [Pop01], Popa proved such a cocycle superrigidity theorem for noncommutative Bernoulli actions of property (T) groups (and more generally, w-rigid groups). Later, in [Pop06], Popa also established cocycle superrigidity for commutative Bernoulli actions $\Lambda \curvearrowright (X, \mu)^\Lambda$ of nonamenable direct product groups Λ, by introducing his spectral gap methods in deformation/rigidity theory. As an appendix, we adapt the proof of [Pop06] to the noncommutative setting and prove a cocycle superrigidity theorem for Connes-Størmer Bernoulli actions of nonamenable direct product groups, hence obtaining the result we need in the proof of Theorems D and E.

Chapter 2

Preliminaries

In this chapter, we recall the type classification of von Neumann algebras and provide further preliminaries that we need for the classification of Bernoulli crossed products. The first section gives an overview of the 'classical' type classification, including the subclassification of type III factors. In Section 2.2, we will introduce the concept of *cocycle actions* and their associated crossed products. Section 2.3 is devoted to Popa's deformation/rigidity theory, and we introduce a class of groups $\mathcal{C}$ for which a certain unique crossed product theorem holds. In Sections 2.4 and 2.5, we discuss structural results of infinite tensor products and general Bernoulli actions.

This chapter contains parts of [VV14] and [Ver15]. Most notably, Sections 2.2, 2.3.3, 2.4 and 2.5 appeared earlier, see Page 50 for more details.

2.1 Type classification of von Neumann algebras

In this section, we give an overview of the type classification for von Neumann algebras, as discovered by Murray and von Neumann [MVN35, MvN36], and we introduce the modular theory of Tomita [Tom67] and Takesaki [Tak70], which allows a subclassification of the type III factors into the classes III_λ, with $\lambda \in [0, 1]$. We conclude the section by introducing Connes's Sd and τ-invariants [Con74], which allows for an even finer subclassification of the type III_1 factors.

2.1.1 Study of projections and types I, II and III

By studying the projections inside a von Neumann algebra, one can classify factors into different types. In this section, we will explain this classification. A *projection* in a von Neumann algebra M is an element $p \in M$ satisfying $p^2 = p = p^\star$. Von Neumann algebras have an abundance of projections: if $x \in M^+$ is a positive element in a von Neumann algebra, i.e. an element of the form $x = y^\star y$ for $y \in M$, then every spectral projection of x is contained in M. Moreover, the set of projections in a von Neumann algebra M is norm dense in M.

An important tool in the type classification, is the comparison of projections. A *partial isometry* in a von Neumann algebra M is an element $v \in M$ such that $v^\star v$ and $vv^\star$ are projections. For $p, q \in M$ projections in a von Neumann algebra, we say that $p \leq q$ if p is a subprojection of q, i.e. if $q - p$ is a projection, and we say that $p \preceq q$ if there exists a partial isometry $v \in M$ such that $p = vv^\star$ and $v^\star v \leq q$. If $p \preceq q$ and $q \preceq p$, then there exists a partial isometry $v \in M$ with $p = vv^\star$ and $q = v^\star v$ (see e.g. [Tak02, Proposition V.1.3]), and we say that p and q are equivalent, which we denote by $p \sim q$. We further denote $p < q$ if $p \leq q, p \neq q$ and $p \prec q$ if $p \preceq q$, $p \not\sim q$.

We say that a nonzero projection $p \in M$ is *minimal* if for all subprojections $q \in M, q \leq p$, q is either zero or equal to p. A projection $p \in M$ is called *infinite* if p is equivalent to a proper subprojection q, i.e. $p \sim q$ with $q < p$, and otherwise p is called *finite*. More generally, we say that a projection $p \in M$ is *purely infinite* if all nonzero subprojections are infinite, and *semifinite* if it has no purely infinite subprojection $q \leq p$. We now can divide factors, i.e. von Neumann algebras with trivial center, into three types. Assume that P is a factor, then P is said to be of type

- I   if P contains a minimal projection,
- II   if P does not contain minimal projections, but contains a nonzero finite projection,
- III   if all nonzero projections in P are infinite.

A factor P is of type I if and only if it is of the form $P \cong B(H)$ for some Hilbert space H, and we can further classify these factors by saying that P is of type I_n, where $n \in \mathbb{N} \cup \{\infty\}$ is the cardinality of an orthonormal basis of H. The type II factors can be divided into two subclasses: we say that a type II factor P is of type II_1 if the unit $1 \in P$ is a finite projection, and otherwise we say that P is of type II_∞.

More generally, one has the following type classification for general von

Neumann algebras. A projection $p \in M$ in a von Neumann algebra M is called *abelian* if pMp is abelian. We call a projection $z \in M$ *central* if $z \in \mathcal{Z}(M)$. Every projection $p \in M$ has a minimal projection $z \in \mathcal{Z}(M)$ such that $p \leq z$, which is called the *central support* of p. We call a von Neumann algebra M *continuous* if M contains no abelian projections, and *discrete* if every nonzero projection $p \in M$ has a nonzero abelian subprojection $q \leq p$. A von Neumann algebra is *semifinite* if all its nonzero projections are semifinite, and *purely infinite* if all nonzero projections are purely infinite. For every von Neumann algebra M, there exists a maximal semifinite central projection $z_{\mathrm{sf}} \in \mathcal{Z}(M)$, and a maximal family of orthogonal abelian projections $p_i \in Mz_{\mathrm{sf}}$, such that for $z_{\mathrm{I}} = \sum_i p_i$, $z_{\mathrm{II}} = z_{\mathrm{sf}} - z_{\mathrm{I}}$ and $z_{\mathrm{III}} = 1 - z_{\mathrm{sf}}$, Mz_{I} is discrete, Mz_{II} is continuous and semifinite, and Mz_{III} is purely infinite. We say that Mz_i is the type i part of M, for $i = \mathrm{I}, \mathrm{II}, \mathrm{III}$, and that M is of type i if $M = Mz_i$.

The above type classification is also reflected in the study of tracial weights. A *weight* on a von Neumann algebra M is a map $\varphi : M^+ \to [0, \infty]$ on the positive cone of M such that $\varphi(x + y) = \varphi(x) + \varphi(y)$ for all $x, y \in M^+$ and $\varphi(\lambda x) = \lambda\varphi(x)$ for all $\lambda \in [0, \infty)$, $x \in M^+$, with the convention that $0 \cdot \infty = 0$. We say that a weight φ on M is

- *normal* if φ is lower semicontinuous, i.e. if for every bounded increasing net $x_i \in M^+$, $\varphi(\sup_i x_i) = \sup_i \varphi(x_i)$;
- *faithful* if $\varphi(x) = 0$ implies that $x = 0$ for all $x \in M^+$;
- *semifinite* if $\{x \in M^+ \mid \varphi(x) < \infty\}$ is σ-strongly dense in M^+; and
- *tracial* if it satisfies $\varphi(x^\star x) = \varphi(xx^\star)$ for all $x \in M$.

A *state* is a weight φ satisfying $\varphi(1) = 1$. For any weight φ, we denote

$$\mathfrak{p}_\varphi = \{x \in M^+ \mid \varphi(x) < \infty\},$$

$$\mathfrak{n}_\varphi = \{x \in M \mid x^\star x \in \mathfrak{p}\},$$

$$\mathfrak{m}_\varphi = \mathrm{span}\{x^\star y \mid x, y \in \mathfrak{n}_\varphi\}.$$

Then $\mathfrak{n}_\varphi$ is a left ideal of M, $\mathfrak{m}_\varphi$ is a $\star$-subalgebra of M such that $\mathfrak{m}_\varphi \cap M^+ = \mathfrak{p}_\varphi$, and every element of $\mathfrak{m}_\varphi$ is a linear combination of four elements in $\mathfrak{p}_\varphi$ [Tak03a, Lemma VII.1.2]. As $\mathfrak{m}_\varphi = \mathrm{span}\,\mathfrak{p}_\varphi$, we will extend a weight φ linearly to a map $\varphi : \mathfrak{m}_\varphi \to \mathbb{C}$. In particular, if φ is a state, it is defined on the whole M. Depending on context, a *trace* will either denote a tracial state (tr, defined on M); or a tracial weight with $\varphi(1) = \infty$ (Tr, defined on M^+).

Consider now any factor P, then P is of type I_n if there exists a normal faithful semifinite tracial weight on P such that the image $\varphi(\mathcal{P}(P))$ of the set $\mathcal{P}(P)$

of projections in P is discrete and has cardinality $n+1$. P is of type II_1 if there exists a normal faithful tracial weight such that $\varphi(\mathcal{P}(P)) = [0,t]$ for some $t \in (0,\infty)$, and P is of type II_∞ if there exists a normal faithful semifinite tracial weight with $\varphi(\mathcal{P}(P)) = [0,\infty]$. Finally, P is of type III if all tracial weights on M are zero. In general, a von Neumann algebra M is *finite* if there exists a normal faithful tracial state on M, and *semifinite* if there exists a normal faithful semifinite tracial weight on M.

In the last part of this section, we construct, for each von Neumann algebra (M,φ) with a weight φ, a 'canonical' Hilbert space on which M acts. Fix a von Neumann algebra M and a normal faithful semifinite weight φ on M, and let $L^2(M,\varphi)$ denote the Hilbert space completion of $\mathfrak{n}_\varphi$ with respect to the inproduct $\langle \cdot,\cdot\rangle_\varphi$ given by $\langle x,y\rangle_\varphi = \varphi(y^\star x)$ for $x,y \in \mathfrak{n}_\varphi$. We will denote the inclusion $\mathfrak{n}_\varphi \subset L^2(M,\varphi)$ by η_φ. If φ is a state, we will also write $\hat{x} = \eta_\varphi(x)$ for any $x \in \mathfrak{n}_\varphi = M$. Note that the left multiplication of M on M is extended to a representation of M on $L^2(M,\varphi)$, by the inequality

$$(ax)^\star(ax) \le \|a\|^2 x^\star x \quad \text{for } a \in M, x \in \mathfrak{n}_\varphi.$$

We denote by $\pi_\varphi : M \to B(L^2(M,\varphi))$ the resulting representation of M, i.e. it is given by $\pi_\varphi(a)\eta_\varphi(x) = \eta_\varphi(ax)$ for $a \in M, x \in \mathfrak{n}_\varphi$. Note that since φ is normal, π_φ is σ-weakly continuous and hence $\pi_\varphi(M)$ is also a von Neumann algebra by Kaplansky's density theorem, and in fact $\pi_\varphi(M) \cong M$. The triplet $\{\pi_\varphi, L^2(M,\varphi), \eta_\varphi\}$ is called the *semi-cyclic* representation of M induced by φ, or the *GNS*-construction, named after Gelfand–Naimark–Segal.

2.1.2 Crossed products by locally compact groups

In this section, we introduce crossed products of actions by locally compact groups, which we will need later to introduce modular theory. Let G be a locally compact group, and M a von Neumann algebra. An action of G on M is a continuous group homomorphism $\alpha : G \to \operatorname{Aut} M$, where the set of automorphisms $\operatorname{Aut} M$ on M carries the topology in which a net of automorphisms α_n converges to $\alpha \in \operatorname{Aut} M$ if and only if $\|\psi \circ \alpha_n - \psi \circ \alpha\|$ converges to zero for every $\psi \in M_\star$. To each action $\alpha : G \curvearrowright M$, we can associate a new von Neumann algebra as follows. Fix a left-invariant Haar measure μ on G, and a *normal faithful* representation $\{\pi, H\}$ of M on a Hilbert space H, i.e. a representation that is σ-weakly continuous and such that $\pi(x) = 0$ implies that $x = 0$ for $x \in M$. Consider the Hilbert space $L^2(G,H)$ of all square integrable H-valued functions (with respect to μ), and define representations π_α of M and λ of G on $L^2(G,H)$ as follows:

$$\big(\pi_\alpha(x)\xi\big)(s) = \pi(\alpha_{s^{-1}}(x))\xi(s), \qquad \text{for } x \in M,\ \xi \in L^2(G,H),$$

$$\big(\lambda(t)\xi\big)(s) = \xi(t^{-1}s), \qquad s,t \in G.$$

The von Neumann algebra generated by $\pi_\alpha(M)$ and $\lambda(G)$ on $L^2(G,H)$ is called the *crossed product* of M by the action α and denoted by $M \rtimes G$. One can show that it is, up to isomorphism, independent of the choice of the normal faithful representation $\{\pi, H\}$ [Tak03a, Theorem X.1.7]. The *group von Neumann algebra* of a locally compact group G is defined as the crossed product of $\mathbb{C}$ by the trivial action of G on $\mathbb{C}$, and we denote $L(G) = \mathbb{C} \rtimes G$.

If G is a discrete group acting on a von Neumann algebra M, and φ is any state on M, it is easy to see that every element $x \in M \rtimes G$ has a *Fourier decomposition* $x = \sum_{g\in G} \pi_\alpha(x_g)\lambda(g)$ with $x_g \in M$, where the sum converges in $L^2(G, L^2(M,\varphi))$. We will now briefly present a second and more intuitive construction of the crossed product for locally compact groups, which is in particular useful when studying generic elements of $M \rtimes G$; replacing the Fourier decomposition for crossed products of discrete groups.

The *σ-strong topology* on $B(H)$ is the topology given by the family of seminorms $x \mapsto \omega(x^\star x)^{\frac{1}{2}}$, where ω ranges over all positive elements in the predual $B_\star(H)$. The *σ-strong** is the weakest topology on $B(H)$ stronger than the σ-strong topology, such that taking the adjoint is continuous. Fix now a locally compact group G with left-invariant Haar measure μ, and an action $\alpha : G \curvearrowright M$ on a von Neumann algebra M with normal faithful representation $\{\pi, H\}$. Denote by $\mathcal{K}(G,M)$ the vector space of all σ-strongly* continuous M-valued functions on G with compact support. Then $\mathcal{K}(G,M)$ becomes an involutive algebra with the following operations, where δ_G is the modular function of G:

$$x * y(t) = \int_G \alpha_s(x(ts))y(s^{-1})d\mu(s), \quad x,y \in \mathcal{K}(G,M),$$

$$x^\sharp(t) = \delta_G(t)^{-1}\alpha_t^{-1}\big(x(t^{-1})^\star\big),$$

and $\mathcal{K}(G,M)$ becomes a two sided M-module by putting $(x\cdot a)(s) = x(s)a$, $(a \cdot x)(s) = \alpha_s^{-1}(a)x(s)$ for $x \in \mathcal{K}(G,M), a \in M$. Then $\mathcal{K}(G,M)$ naturally acts on $L^2(G,H)$ by putting

$$\big(\tilde{\pi}_\alpha(x)\xi\big)(s) = \int_G \pi(\alpha_t(x(st)))\xi(t^{-1})dt, \quad \text{for } \xi \in L^2(G,H),$$

and $\tilde{\pi}_\alpha$ is moreover a $\star$-representation of $\mathcal{K}(G,M)$, as $\tilde{\pi}_\alpha(x^\sharp) = \tilde{\pi}_\alpha(x)^\star$ and $\tilde{\pi}_\alpha(x * y) = \tilde{\pi}_\alpha(x)\tilde{\pi}_\alpha(y)$ for $x,y \in \mathcal{K}(G,M)$. Then $\mathcal{K}(G,M)$ generates the crossed product: $M \rtimes G \cong \tilde{\pi}_\alpha(\mathcal{K}(G,M))''$.

2.1.3 Duality for crossed products by abelian groups

In this short section, we present an important duality result for actions of abelian groups on von Neumann algebras.

Let M be a von Neumann algebra, and G a locally compact abelian group that acts on M via $\alpha : G \curvearrowright M$. Consider a normal faithful representation $\{\pi, H\}$ of M on a Hilbert space H, and recall that the crossed product $M \rtimes G$ is generated by the representations π_α of M and λ of G, on the Hilbert space $L^2(G, H)$ of all square integrable H-valued functions on G.

Denote now by $\hat{G}$ the Pontryagin dual of G, and denote by $\langle s, p \rangle$, $s \in G, p \in \hat{G}$ the pairing of G and $\hat{G}$. We now represent $\hat{G}$ on $L^2(G, H)$ by putting

$$\big(\mu(p)\xi\big)(s) = \overline{\langle s, p \rangle}\xi(s), \quad \text{for } s \in G,\ p \in \hat{G},\ \xi \in L^2(G, H).$$

Then the representation μ induces an action $\hat{\alpha}$ of $\hat{G}$ on $M \rtimes_\alpha G$, defined as follows:

$$\hat{\alpha}_p(x) = \mu(p)x\mu(p)^\star, \quad x \in M \rtimes G,\ p \in \hat{G}.$$

Indeed, one checks easily that $\hat{\alpha}_p(\pi_\alpha(x)) = \pi_\alpha(x)$ for $x \in M$ and that $\hat{\alpha}_p(\lambda(s)) = \overline{\langle s, p \rangle}\lambda(s)$ for $s \in G$.

The following theorem states that taking the double crossed product, first by α and then by $\hat{\alpha}$, yields again the original von Neumann algebra M, up to scaling.

Theorem 2.1 (Takesaki duality, [Tak03a, Theorem X.2.3])**.** *Let M a von Neumann algebra with normal faithful representation $\{\pi, H\}$, and let $\alpha : G \curvearrowright M$ be an action of a locally compact abelian group on M. Denote by $\hat{\alpha} : \hat{G} \curvearrowright M \rtimes_\alpha G$ the dual action. The following statements hold true:*

1. *The fixed point algebra of $M \rtimes_\alpha G$ under $\hat{\alpha}$ is precisely $\pi_\alpha(M)$.*
2. *There exists a unique isomorphism Φ of $(M \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}$ onto $M \overline{\otimes} B(L^2(G))$ such that*

$$\begin{aligned} \{\Phi\big(\pi_{\hat{\alpha}} \circ \pi_\alpha(x)\big)\xi\}(s) &= (\alpha_t)^{-1}(x)\xi(s), \quad && x \in M, \quad \xi \in L^2(G, H), \quad s \in G, \\ \{\Phi\big(\pi_{\hat{\alpha}} \circ \lambda(t)\big)\xi\}(s) &= \xi(s - t), && t \in G, \\ \{\Phi\big(\lambda(p)\big)\xi\}(s) &= \overline{\langle s, p \rangle}\xi(s), && p \in \hat{G}. \end{aligned} \tag{2.1}$$

   Here $(\lambda(p))_{p \in \hat{G}}$ is the canonical group of unitaries in the second crossed product.

3. *The bidual action* $\hat{\hat{\alpha}} : G \curvearrowright (M \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}$ *corresponds under* Φ *to the action* $(\alpha_s \otimes \operatorname{Ad}\lambda(s)^\star)_{s\in G}$ *on* $M \mathbin{\overline{\otimes}} B(L^2(G))$, *where* $\lambda(s)$ *denotes the left regular representation of* G *on* $B(L^2(G))$ *given by* $(\lambda(s)f)(t) = f(t-s)$ *for* $f \in L^2(G)$.

2.1.4 Tomita-Takesaki's modular theory for type III factors

The existence of a faithful trace turned out to be crucial for the study of type II_1 factors, and the absence thereof in the type III setting makes the analysis of type III factors much harder. The modular theory of Tomita and Takesaki, providing a framework to study type III factors in more detail, was therefore a major breakthrough. In this section, we will introduce this modular theory, and we will show how it allows us to provide a finer classification of type III factors. Modular theory assigns to any normal faithful weight φ on a von Neumann algebra M, a certain *modular action* $\mathbb{R} \curvearrowright M$.

Fix a von Neumann algebra M and a normal semifinite faithful (n.s.f.) weight φ on M, and consider the GNS-representation of M on $L^2(M,\varphi)$. The involution $\star$ on M defines an unbounded operator $S_0 : L^2(M,\varphi) \to L^2(M,\varphi)$ with domain $\mathfrak{D}(S_0) = \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^\star$, given by $x \mapsto x^\star$ for all $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^\star$.

Lemma 2.2 (See proof of [Tak03a, Theorem VII.2.6])**.** *The unbounded operator* $S_0 : L^2(M,\varphi) \to L^2(M,\varphi)$ *with domain* $\mathfrak{D}(S_0) = \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^\star$, *given by* $x \mapsto x^\star$ *for all* $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^\star$, *is preclosed.*

We denote the closure of S_0 by S_φ. The *modular operator* Δ_φ of the n.s.f. weight φ is the linear positive self-adjoint operator $\Delta_\varphi = S_\varphi^\star S_\varphi$. Note that the domain $\mathfrak{D}(\Delta_\varphi) \subset L^2(M,\varphi)$ is dense. Let $S_\varphi = J_\varphi(\Delta_\varphi)^{\frac{1}{2}}$ be the polar decomposition of S_φ. Then $J_\varphi \in B(L^2(M,\varphi))$ is an isometry satisfying $J_\varphi^2 = 1$, we call it the *modular conjugation*. The following theorem plays an important role in modular theory.

Theorem 2.3 ([Tak03a, Theorem VI.1.19])**.** *Let* (M,φ) *be a von Neumann algebra with an n.s.f. weight* φ, *and denote by* Δ_φ *and* J_φ *the modular operator and the modular conjugation. Then we have that, as subalgebras of* $B(L^2(M,\varphi))$,

$$J_\varphi \pi_\varphi(M) J_\varphi = \pi_\varphi(M)', \qquad \Delta_\varphi^{\mathbf{i}t} \pi_\varphi(M) \Delta_\varphi^{-\mathbf{i}t} = \pi_\varphi(M) \quad \textit{for all } t \in \mathbb{R}.$$

The action σ^φ of $\mathbb{R}$ on $M \cong \pi_\varphi(M)$ given by $\sigma_t^\varphi(x) = \Delta_\varphi^{\mathbf{i}t} x \Delta_\varphi^{-\mathbf{i}t}$ for $t \in \mathbb{R}$, $x \in \pi_\varphi(M)$, is called the *modular action*; and the one-parameter group of automorphisms $\sigma_{t\in\mathbb{R}}^\varphi$ is the *modular automorphism group*. The following theorem provides a characterization of the modular automorphism group.

Theorem 2.4 ([Tak03a, Theorem VIII.1.2])**.** *Let (M, φ) be a von Neumann algebra with a n.s.f. weight φ. The modular automorphism group $\sigma^\varphi_{t\in\mathbb{R}}$ is the unique continuous one parameter automorphism group $\alpha_{t\in\mathbb{R}}$ satisfying the following two conditions:*

(i) $\varphi = \varphi \circ \alpha_t$, *for all* $t \in \mathbb{R}$*;*

(i) *For every pair* $x, y \in \mathfrak{n}_\varphi \cap \mathfrak{n}^\star_\varphi$*, there exists a bounded continuous function* F *on the strip* $\{z \in \mathbb{C} \mid 0 \leq \Im z \leq 1\}$ *which is analytic on the interior* $\{z \in \mathbb{C} \mid 0 < \Im z < 1\}$*, such that*

$$F(t) = \varphi(\alpha_t(x)y) \quad \textit{and} \quad F(t + \mathbf{i}) = \varphi(y\alpha_t(x)) \quad \textit{for all } t \in \mathbb{R}.$$

The above characterization allows to first 'guess' what the modular automorphism group is, and then check that conditions (i) and (ii) are fullfilled. As a consequence of Theorem 2.4, we can also give a nice characterization of the *centralizer* M_φ of M with respect to the weight φ. The centralizer is defined as the fixed point set of the modular action, $M_\varphi = \{x \in M \mid \sigma^\varphi_t(x) = x \text{ for all } t \in \mathbb{R}\}$, and using Theorem 2.4, it follows that

$$M_\varphi = \{a \in M \mid a\mathfrak{m}_\varphi \subset \mathfrak{m}_\varphi, \mathfrak{m}_\varphi a \subset \mathfrak{m}_\varphi \text{ and } \varphi(ax) = \varphi(xa) \text{ for all } x \in \mathfrak{m}_\varphi\},$$

see e.g. [Tak03a, Theorem VIII.2.6]. In particular, the center of a von Neumann algebra is always contained in the centralizer, and a state φ is tracial if and only if $\sigma^\varphi_t = \mathrm{id}$ for all $t \in \mathbb{R}$.

The following theorem shows the real power of the modular action. It allows to study III factors through studying the more 'familiar' semifinite von Neumann algebras. Let (M, φ) be a von Neumann algebra with a n.s.f. weight φ, and denote by $M \rtimes \mathbb{R}$ the crossed product of M with the modular action σ^φ of the weight φ. Consider $\mathbb{R}^+_0$ to be the dual of $\mathbb{R}$ under the pairing $\langle t, \mu\rangle = \mu^{\mathbf{i}t}$ for $t \in \mathbb{R}, \mu \in \mathbb{R}^+_0$, and let $\theta : \mathbb{R}^+_0 \curvearrowright M \rtimes \mathbb{R}$ denote the dual action, i.e. $\theta_\mu\big(\pi_{\sigma^\varphi}(x)\lambda_\varphi(t)\big) = \overline{\langle t, \mu\rangle}\pi_{\sigma^\varphi}(x)\lambda_\varphi(t)$ for $x \in M, t \in \mathbb{R}, \mu \in \mathbb{R}^+_0$.

Theorem 2.5 (See proof of [Tak03a, Theorem XII.1.1 (i)])**.** *The crossed product $M \rtimes \mathbb{R}$ with the modular action of φ, carries a semi-finite normal faithful tracial weight* Tr_φ *such that*

$$\mathrm{Tr}_\varphi \circ\theta_\mu = \mu^{-1}\,\mathrm{Tr}_\varphi \quad \textit{for all } \mu \in \mathbb{R}^+_0,$$

and such that the dual weight $\tilde{\mathrm{Tr}}_\varphi$ *on* $(M \rtimes \mathbb{R}) \rtimes \mathbb{R}^+_0$*, which is defined by*

$$\tilde{\mathrm{Tr}}_\varphi\big(\tilde\pi_\theta(f)^\star\tilde\pi_\theta(f)\big) = \mathrm{Tr}_\varphi\left((f^\sharp * f)(e)\right) \quad \textit{for } f \in \mathcal{K}(\mathbb{R}^+_0, M \rtimes \mathbb{R}) \cdot \mathfrak{n}_{\mathrm{Tr}_\varphi},$$

$$\sigma^{\tilde{\mathrm{Tr}}_\varphi}_t\big(\pi_\theta(x)\lambda(\mu)\big) = \pi_\theta(x)\lambda(\mu)\mu^{-\mathbf{i}t} \qquad \textit{for } x \in M \rtimes \mathbb{R},\ \mu \in \mathbb{R}^+_0,\ t \in \mathbb{R},$$

corresponds under the Takesaki duality to the weight $\varphi \otimes \mathrm{Tr}(h\,\cdot)$ *on* $M \mathbin{\overline{\otimes}} B(L^2(\mathbb{R}))$.

Here h *is the hermitian operator given by* $h^{\mathbf{i}t} = \lambda(t) \in B(L^2(\mathbb{R}))$, *with* $\lambda(t)$ *the left regular representation defined by* $\big(\lambda(t)f\big)(s) = f(s-t)$ *for* $f \in L^2(\mathbb{R})$.

Another result which shows the importance of the modular action, is the following existence result of *conditional expectations* onto subalgebras. Let $M \subset N$ be a unital inclusion of von Neumann algebras, and φ a normal semifinite faithful weight on N such that the restriction $\varphi|_M$ of φ to M is semifinite. A linear map $E : N \to M$ is called the *conditional expectation of N onto M with respect to φ* if the following conditions are satisfied:

$$\|E(x)\| \leq \|x\| \quad \text{for all } x \in M, \quad E(x) = x \quad \text{for all } x \in N, \quad \text{and } \varphi = \varphi \circ E.$$

A conditional expectation $E : N \to M$ automatically satisfies $E(x^\star x) \geq 0$, $E(axb) = aE(x)b$ and $E(x)^\star E(x) \leq E(x^\star x)$ for all $x \in N, a, b \in M$ [Tak02, Theorem III.3.4].

Theorem 2.6 ([Tak03a, Theorem IX.4.2])**.** *Let* $M \subset N$ *be a unital inclusion of von Neumann algebras, and let* φ *be a normal semifinite faithful weight on* N *such that the restriction* $\varphi|_M$ *on* M *is semifinite. There exists a conditional expectation* $E : N \to M$ *with respect to* φ *if and only if* M *is globally invariant under the modular action of* φ, $\sigma_t^\varphi(M) = M$ *for all* $t \in \mathbb{R}$. *If this is the case, the conditional expectation* E *is normal and uniquely determined by* φ.

If now φ and ψ are different n.s.f. weights on the same von Neumann algebra M, one obtains two modular actions $\sigma^\varphi : \mathbb{R} \curvearrowright M$ and $\sigma^\psi : \mathbb{R} \curvearrowright M$. However, these actions are still cocycle conjugate, as stated in the next theorem. Recall that the σ-strong topology on $B(H)$ is the topology given by the family of seminorms $x \mapsto \omega(x^\star x)^{\frac{1}{2}}$, where ω ranges over all positive elements in the predual $B_\star(H)$.

Theorem 2.7 (Connes's cocycle derivative theorem, [Tak03a, Theorem VIII.3.3])**.** *Let* M *be a von Neumann algebra with n.s.f. weights* φ *and* ψ. *Then there exists a unique* σ*-strongly continuous one parameter family* $u_{t\in\mathbb{R}}$ *of unitaries in* M *such that:*

(i) $u_{s+t} = u_s \sigma_s^\psi(u_t)$ *for* $s, t \in \mathbb{R}$;

(i) $\sigma_t^\varphi(x) = u_t \sigma_t^\psi(x) u_t^\star$ *for* $x \in M, t \in \mathbb{R}$;

(i) *For any* $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\psi^\star, y \in \mathfrak{n}_\psi \cap \mathfrak{n}_\varphi^\star$, *there exists a bounded continuous function* F *on the strip* $\{z \in \mathbb{C} \mid 0 \leq \Im z \leq 1\}$ *which is analytic on the interior* $\{z \in \mathbb{C} \mid 0 < \Im z < 1\}$, *such that*

$$F(t) = \varphi(u_t \sigma_t^\psi(y) x), \quad \textit{and} \quad F(t + \mathbf{i}) = \psi(x u_t \sigma_t^\psi(y)) \quad \textit{for all } t \in \mathbb{R}.$$

In particular, we get that for any two n.s.f. weights φ and ψ, the crossed products $M \rtimes_{\sigma^\varphi} \mathbb{R}$ and $M \rtimes_{\sigma^\psi} \mathbb{R}$ are isomorphic, as $\pi_{\sigma^\varphi}(x)\lambda_\varphi(t) \mapsto \pi_{\sigma^\psi}(x)u_t\lambda_\psi(t)$ for $x \in M, t \in \mathbb{R}$ implements an isomorphism. Furthermore, the dual actions $\mathbb{R}_0^+ \curvearrowright M \rtimes_{\sigma^\varphi} \mathbb{R}$ and $\mathbb{R}_0^+ \curvearrowright M \rtimes_{\sigma^\psi} \mathbb{R}$ are conjugate under this isomorphism. This is the motivation to study the object '$M \rtimes \mathbb{R}$' and the action '$\mathbb{R}_0^+ \curvearrowright M \rtimes \mathbb{R}$' abstractly, without reference to the weight[1]. We call $M \rtimes \mathbb{R}$ the *continuous core*, and $\mathbb{R}_0^+ \curvearrowright \mathcal{Z}(M \rtimes \mathbb{R})$ the *flow of weights*, following [Tak03a, Definition XII.1.3].

We are now ready to introduce a finer classification of the type III factors. Let P be a type III factor, and let $\mathbb{R}_0^+ \curvearrowright \mathcal{Z}(N)$ denote the flow of weights, with N the continuous core of P. The flow of weights is then ergodic, see [Tak03a, Corollary XII.1.4]. We say that P is of type

- III_1 if N is a factor, i.e. the flow of weights is the trivial action on a point,
- $\mathrm{III}_{\lambda\in(0,1)}$ if $\mathbb{R}_0^+ \curvearrowright \mathcal{Z}(N)$ is periodic[2] with period λ^{-1},
- III_0 if $\mathbb{R}_0^+ \curvearrowright \mathcal{Z}(N)$ is not periodic.

We conclude this section with the following well known lemma, which provides a description of the relative commutant of the copy of $\mathbb{R}$ inside the continuous core. A proof can for example be found in [HR10, Proposition 2.4], but for the sake of completeness, we also include a proof here.

Lemma 2.8. *Let (M, φ) be a von Neumann algebra equipped with a n.s.f. weight. Then $(L\mathbb{R})' \cap M \rtimes_{\sigma^\varphi} \mathbb{R} = M_\varphi \mathbin{\overline{\otimes}} L\mathbb{R}$.*

Proof. Put $N = M \rtimes_{\sigma^\varphi} \mathbb{R}$, and consider the Takesaki duality Θ for the action σ^φ as in (2.1). Note that $\Theta(L\mathbb{R}) = 1 \otimes L\mathbb{R}$ and that $\Theta(N) \subset M \mathbin{\overline{\otimes}} B(L^2(\mathbb{R}))$ is the fixed point set of the action $(\sigma_s^\varphi \otimes \operatorname{Ad}\lambda(s)^\star)_{s\in\mathbb{R}}$. The former implies that $\Theta((L\mathbb{R})' \cap N) \subset M \mathbin{\overline{\otimes}} L\mathbb{R}$, which combined with the latter yields that $\Theta((L\mathbb{R})' \cap N) \subset (M \mathbin{\overline{\otimes}} L\mathbb{R})^{\sigma^\varphi \otimes \mathrm{id}}$, as the action $(\operatorname{Ad}\lambda(s))_{s\in\mathbb{R}}$ is trivial on $L\mathbb{R}$. But it is now easy to see that $(M \mathbin{\overline{\otimes}} L\mathbb{R})^{\sigma^\varphi \otimes \mathrm{id}} = M_\varphi \mathbin{\overline{\otimes}} L\mathbb{R} = \Phi(M_\varphi \mathbin{\overline{\otimes}} L\mathbb{R})$, proving the inclusion $(L\mathbb{R})' \cap N \subset M_\varphi \mathbin{\overline{\otimes}} L\mathbb{R}$. The other inclusion being obvious, this completes the proof. □

[1] However note that in this thesis, the position of $L\mathbb{R}$ inside $M \rtimes_{\sigma^\varphi} \mathbb{R}$ actually plays a crucial role (see Chapter 4). For this reason, we will always explicitly choose a n.s.f. weight and study the crossed product for its modular action.

[2] A nontrivial action $\alpha : \mathbb{R}_0^+ \curvearrowright M$ is periodic with period $\mu > 1$ if $\alpha_\mu = \mathrm{id}$ and if there exists no μ', $1 < \mu' < \mu$, such that $\alpha_{\mu'} = \mathrm{id}$.

2.1.5 Connes's invariants for type III_1 factors

To study factors of type III_1, Connes [Con74] introduced the invariants Sd and τ, which we recall here. A normal semifinite faithful (n.s.f.) weight φ on a von Neumann algebra M is called *almost periodic* if the modular operator Δ_φ on $L^2(M,\varphi)$ is diagonalizable. Let M be a factor, then the *point modular spectrum* is the subset of $\mathbb{R}_0^+$ defined by

$$\operatorname{Sd}(M) = \bigcap_{\psi \text{ almost periodic weight on } M} \text{point spectrum}\, \Delta_\psi .$$

A factor M is *full* when $\operatorname{Inn}(M)$, the set of inner automorphisms $\operatorname{Ad} u$ for $u \in M$, is closed inside $\operatorname{Aut}(M)$. If M is a full factor with separable predual, and if φ is a normal faithful almost periodic weight on M, then $\operatorname{Sd}(M) =$ point spectrum Δ_φ if and only if M_φ is a factor (see [Con74, Lemma 4.8]). An arbitrary factor M is full if and only if every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in M such that $\forall \psi \in M_\star : \|\psi(x_n\,\cdot) - \psi(\cdot\, x_n)\| \to 0$, is trivial, i.e. $x_n - z_n 1 \to 0$ $\star$-strongly for some sequence $(z_n)_{n\in\mathbb{N}} \in \mathbb{C}$ (see [Con74, Theorem 3.1]). In particular, if all central sequences in M are trivial, M is full (see [Con74, Corollary 3.7]). A bounded sequence $(x_n)_{n\in\mathbb{N}}$ is called *central* if for all $y \in M$, $x_n y - y x_n \to 0$ $\star$-strongly.

Let M now be a full factor of type III_1. Denote by $\pi : \operatorname{Aut} M \to \operatorname{Out} M$ the canonical projection to $\operatorname{Out} M := \operatorname{Aut} M / \operatorname{Inn} M$. Let ψ be any weight on M, and define $\delta : \mathbb{R} \to \operatorname{Out} M$ by putting $\delta(t) = \pi(\sigma_t^\psi)$. This map is independent of the choice of the weight ψ, due to Connes's Radon-Nikodym cocycle theorem (see Theorem 2.7). The τ invariant of M, denoted by $\tau(M)$, is the weakest topology on $\mathbb{R}$ that makes δ continuous, with respect to the quotient topology on $\operatorname{Out}(M)$.

2.2 Cocycle actions

In this section, we mainly introduce the needed notation and terminology for studying cocycle actions, further extending Section 2.1.2.

2.2.1 Definitions

Let M_1 and M_2 be von Neumann algebras equipped with normal semifinite faithful (n.s.f.) weights φ_1 and φ_2 respectively. A homomorphism $\alpha : M_1 \to M_2$

is called *weight scaling* if $\varphi_2 \circ \alpha = \lambda\varphi_1$ for some $\lambda \in \mathbb{R}_0^+$. The number λ is called the *modulus* of α, and will be denoted as $\operatorname{mod}_{\varphi_1,\varphi_2} \alpha$. We denote by $\operatorname{Aut} M$ the group of automorphisms of M, and by $\operatorname{Aut}(M,\varphi)$ the subgroup of weight scaling automorphisms of M. We equip $\operatorname{Aut} M$ with the topology where a net α_n of automorphisms of M converges to $\alpha \in \operatorname{Aut} M$ if and only if for every $\psi \in M_\star$, $\|\psi \circ \alpha_n - \psi \circ \alpha\|$ converges to zero. With this topology, the groups $\operatorname{Aut} M$ and $\operatorname{Aut}(M,\varphi)$ become Polish groups [Ara72, Haa73].

Let G be a locally compact group, and (M,φ) a von Neumann algebra with an n.s.f. weight. We denote the *centralizer* of φ by M_φ, i.e. $M_\varphi = \{u \in \mathcal{U}(M) \mid \forall x \in M^+ : \varphi(x) = \varphi(uxu^\star)\}''$. A *cocycle action* of G on (M,φ) is a continuous map $\alpha : G \to \operatorname{Aut}(M,\varphi) : g \mapsto \alpha_g$ and a continuous map $v : G \times G \to \mathcal{U}(M_\varphi)$ such that:

$$\alpha_e = \mathrm{id}, \qquad \alpha_g \circ \alpha_h = \operatorname{Ad} v_{g,h} \circ \alpha_{gh}, \qquad \forall g,h,k \in G,$$
$$v_{g,h}v_{gh,k} = \alpha_g(v_{h,k})v_{g,hk},$$
$$v_{1,g} = v_{g,1} = 1.$$

Such an action is denoted by $G \curvearrowright^{\alpha,v} (M,\varphi)$. A strongly continuous map v satisfying the above relations is called a 2-*cocycle* for α. If $v = 1$, α is called an *action*.

Two cocycle actions $G \curvearrowright^{\alpha,v} (M_1,\varphi_1)$ and $G \curvearrowright^{\beta,w} (M_2,\varphi_2)$ are *cocycle conjugate through a weight scaling isomorphism* if there exists a strongly continuous map $u : G \to \mathcal{U}((M_2)_{\varphi_2})$ and a weight scaling isomorphism $\psi : (M_1,\varphi_1) \to (M_2,\varphi_2)$ satisfying

$$\psi \circ \alpha_g = \operatorname{Ad} u_g \circ \beta_g \circ \psi, \qquad \forall g,h \in G, \tag{2.2}$$
$$\psi(v_{g,h}) = u_g\beta_g(u_h)w_{g,h}u_{gh}^\star. \tag{2.3}$$

If moreover $\operatorname{mod}_{\varphi_1,\varphi_2} \psi = 1$, we say that (α,v) and (β,w) are cocycle conjugated *through a weight preserving isomorphism*. If α, β are actions and (2.2) holds for $u = 1$, we say that α and β are *conjugate*. If $\delta : G_1 \to G_2$ is a continuous isomorphism, we say that the cocycle actions $G_1 \curvearrowright^{\alpha,v} (M_1,\varphi_1)$ and $G_2 \curvearrowright^{\beta,w} (M_2,\varphi_2)$ are cocycle conjugate, resp. conjugate, *modulo* δ, if the actions α and $\beta \circ \delta$ of G_1 are cocycle conjugate, resp. conjugate.

Let G be a locally compact group, (M,φ) a von Neumann algebra equipped with an n.s.f. weight, and $G \curvearrowright^{\alpha,v} (M,\varphi)$ a cocycle action of G on M. We associate the following new von Neumann algebra to the cocycle action α, v. Consider the Hilbert space $L^2(G, L^2(M,\varphi))$ of all square integrable $L^2(M,\varphi)$-valued functions (with respect to a fixed left invariant Haar measure μ on G, see also Section 2.1.2), and define a representation π_α of M and unitaries $\lambda_v(g)$ for $g \in G$ on $L^2(G, L^2(M,\varphi))$ as follows:

$$\big(\pi_\alpha(x)\xi\big)(s) = \pi(\alpha_{s^{-1}}(x))\xi(s), \qquad \text{for } x \in M,\ \xi \in L^2(G,H),$$

$$\big(\lambda_v(t)\xi\big)(s) = \pi(v_{s^{-1},t})\xi(t^{-1}s), \qquad s,t \in G.$$

The von Neumann algebra generated by $\pi_\alpha(M)$ and $\lambda_v(G)$ is called the *crossed product* of M by the cocycle action α, v and denoted by $M \rtimes_{\alpha,v} G$. Note that we have the relations $\pi_\alpha(\alpha_s(x)) = \lambda_v(s)\pi_\alpha(x)\lambda_v(s)^\star$ and $\lambda_v(s)\lambda_v(t) = \pi_\alpha(v_{s,t})\lambda_v(st)$.

2.2.2 Properties of cocycle actions

In this subsection, we define two properties that cocycle actions can have.

An automorphism α of M is called *properly outer* if there exists no nonzero element $y \in M$ such that $y\alpha(x) = xy$ for all $x \in M$. A cocycle action $\Gamma \curvearrowright^{\alpha,v} (M,\varphi)$ of a discrete group on a von Neumann algebra with an n.s.f. weight is called *properly outer* if α_g is properly outer for all $g \in \Gamma, g \neq e$. The importance of properly outer actions is captured in the following easy lemma.

Lemma 2.9 (Folklore). *Let (M,φ) be a von Neumann algebra equipped with a normal faithful state φ, and let Γ be a countable discrete group that acts on M via a state-preserving cocycle action $\Gamma \curvearrowright^{\alpha,v} (M,\varphi)$. Denote by $\Gamma_0 < \Gamma$ the subgroup of elements with finite conjugacy classes. The following statements hold true.*

(i) $\{\lambda(s) \mid s \in \Gamma\}' \cap (M \rtimes \Gamma) \subset (M \rtimes \Gamma_0)$.

(ii) *If the action $\Gamma_0 \curvearrowright M$ is properly outer, then*

$$\mathcal{Z}(M \rtimes \Gamma) = \{x \in \mathcal{Z}(M) \mid \alpha_s(x) = x \text{ for all } s \in \Gamma\}.$$

More generally, if $\Gamma_1 < \Gamma_0$ is a subgroup such that for all $g \in \Gamma_0 - \Gamma_1$, α_g is properly outer, then $\mathcal{Z}(M \rtimes \Gamma) \subset M \rtimes \Gamma_1$.

Proof. (i) Let $x \in \{\lambda(s) \mid s \in \Gamma\}' \cap (M \rtimes \Gamma)$, and write $x = \sum_{s\in\Gamma} x_s\lambda(s)$ as a $\|\cdot\|_2$-converging sum for $x_s \in M$. The fact that x commutes with $\lambda(t)$ for all $t \in \Gamma$ implies that $\alpha_t(x_s)v_{t,s} = x_{tst^{-1}}v_{tst^{-1},t}$ for all $s,t \in \Gamma$, and in particular $\|\alpha_t(x_s)\|_2 = \|x_{tst^{-1}}\|_2$. As the set $\{tst^{-1} \mid t \in \Gamma\}$ is infinite for all $s \in \Gamma - \Gamma_0$, it follows that $x_s = 0$ for all $s \in \Gamma - \Gamma_0$, hence $x \in M \rtimes \Gamma_0$.

(ii) Assume now that $x \in \mathcal{Z}(M \rtimes \Gamma)$ and that $\Gamma_1 < \Gamma_0$ is a possibly trivial subgroup such that α_g is properly outer for all $g \in \Gamma_0 - \Gamma_1$. By (i), we can write $x = \sum_{s\in\Gamma_0} x_s\lambda(s)$ for $x_s \in M$. Since x commutes with M, we have for all $y \in M$ and for all $s \in \Gamma_0$ that $yx_s = x_s\alpha_s(y)$. Since $\Gamma_0 - \Gamma_1$ acts properly outerly, this implies that $x_s = 0$ for all $s \in \Gamma_0 - \Gamma_1$, thus $x \in M \rtimes \Gamma_1$. Now suppose that $\Gamma_1 = \{1\}$, then $x \in M$. As x commutes with all $\lambda(t)$ for $t \in \Gamma$, it furthermore follows that $\alpha_s(x) = x$ for all $s \in \Gamma$. This shows that $\mathcal{Z}(M \rtimes \Gamma)$ is a subset of the fixed points in M under α, the other inclusion is obvious. □

Let now $G \curvearrowright^\alpha (M, \varphi)$ be a state-preserving genuine action of a locally compact group G on a von Neumann algebra M with a normal faithful state. We say that the action α is *weakly mixing* if $L^2(M, \varphi)$ does not admit finite-dimensional invariant subspaces under α other than $\mathbb{C}\hat{1}$. The following lemma gives another characterization of weakly mixing actions.

Lemma 2.10 (See e.g. [Pop01, Proposition 2.4.2]). *Let G be a locally compact group, (M, φ) a von Neumann algebra with a normal faithful state, and $G \curvearrowright^\alpha (M, \varphi)$ a state-preserving action. The following two statements are equivalent.*

(i) *The action $G \curvearrowright^\alpha (M, \varphi)$ is weakly mixing.*

(ii) *For any finite subset $\mathcal{F} \subset M$ and any $\epsilon > 0$, there exists $g \in G$ such that*

$$|\varphi(\alpha_g(x)y) - \varphi(x)\varphi(y)| < \epsilon \quad \text{for all } x, y \in \mathcal{F}.$$

Note that in the case $G \curvearrowright^\alpha (M, \varphi)$ is an action of a discrete group, the result is exactly the content of [Pop01, Proposition 2.4.2]. If $G \curvearrowright^\alpha (M, \varphi)$ is an action of a locally compact group G, then choose a dense countable subgroup $\Gamma \subset G$ and note that both statements hold for G if and only if they hold for Γ, as actions are required to be continuous.

2.2.3 Reductions of cocycle actions of discrete groups

In this final subsection, we show how one can reduce a properly outer cocycle action of a countable discrete group $\Gamma \curvearrowright (M, \varphi)$ with a projection in M_φ.

Assume that (M, φ) is a von Neumann algebra with an n.s.f. weight for which M_φ is a factor. Consider a properly outer cocycle action $\Gamma \curvearrowright^{\alpha,v} (M, \varphi)$ of a discrete group Γ, that preserves the weight φ. Suppose that $p \in M_\varphi$ is a nonzero projection with $\varphi(p) < \infty$, and choose partial isometries $w_g \in M_\varphi$ such that $p = w_g w_g^\star$, $\alpha_g(p) = w_g^\star w_g$, $w_e = p$, which exist since $(p \vee \alpha_g(p))M_\varphi(p \vee \alpha_g(p))$ is a finite factor for all $g \in \Gamma$. Let $\alpha^p : \Gamma \to \operatorname{Aut}(pMp, \varphi_p)$ be defined by

$\alpha^p_g(pxp) = w_g\alpha_g(pxp)w_g^\star$, for $x \in M$, where $\varphi_p(pxp) = \varphi(pxp)/\varphi(p)$. Denote $v^p_{g,h} = w_g\alpha_g(w_h)v_{g,h}w^\star_{gh} \in pM_\varphi p$, for $g, h \in \Gamma$. Then $\Gamma \curvearrowright^{\alpha^p,v^p} (pMp, \varphi_p)$ is a properly outer cocycle action, which does not depend on the choice of the partial isometries $w_g \in M_\varphi$, up to state-preserving cocycle conjugacy. We call this action the *reduced cocycle action of* α *by* p.

We say that two properly outer cocycle actions $\Gamma \curvearrowright^{\alpha,v} (M_1, \varphi_1)$ and $\Gamma \curvearrowright^{\beta,w} (M_2, \varphi_2)$ of a discrete group Γ are, *up to reductions*, cocycle conjugate through a state-preserving isomorphism, if there exists projections $p_i \in (M_i)_{\varphi_i}$ such that the reduced cocycle actions α^{p_0} and β^{p_1} are cocycle conjugate through a state-preserving isomorphism.

We end this subsection with the following related definition. Let $\alpha : \Gamma \to \mathrm{Aut}(M, \varphi)$ be an action of a locally compact group on a von Neumann algebra (M, φ) with an n.s.f. weight φ. A *generalized 1-cocycle* for α with *support projection* $p \in M_\varphi$ is a continuous map $w : \Gamma \to M_\varphi$ such that $w_g \in pM_\varphi\alpha_g(p)$ is a partial isometry with $p = w_g w_g^\star$ and $\alpha_g(p) = w_g^\star w_g$, and

$$w_{gh} = \Omega(g, h)w_g\alpha_g(w_h) \quad \text{for all } g, h \in \Gamma,$$

where $\Omega(g, h)$ is a scalar 2-cocycle.

2.3 Uniqueness of regular amenable subalgebras and a class $\mathcal{C}$ of groups

In Chapters 3 and 4 below, we show that Bernoulli crossed products $P^\Lambda \rtimes \Lambda$ remember the group Λ, by first passing to the discrete or continuous core, and then applying the main results of [PV11, PV12, Ioa12] providing unique crossed product decomposition theorems for factors of the form $R \rtimes \Lambda$, where $\Lambda \curvearrowright R$ is an outer action on the hyperfinite II_1 factor R of a group Λ that is a free group, or a hyperbolic group, or a free product group, etc. In this section, we introduce the class $\mathcal{C}$ of countable groups Λ for which such a unique crossed product decomposition theorem holds. We prove that this class $\mathcal{C}$ is stable under taking extensions and stable under commensurability.

2.3.1 Popa's theory of intertwining-by-bimodules

We recall from [Pop03] Popa's theory of intertwining-by-bimodules.

Theorem 2.11. *[Pop03, Theorem 2.1 and Corollary 2.3] Let (M, τ) be any tracial von Neumann algebra, and let $P \subset 1_P M 1_P$ and $Q \subset 1_Q M 1_Q$ be von Neumann subalgebras. The following three statements are equivalent.*

(1) *There exist a projection* $p \in M_n(\mathbb{C}) \otimes Q$, *a normal unital* $\star$*-homomorphism* $\theta : P \to p(M_n(\mathbb{C}) \otimes Q)p$ *and a non-zero partial isometry* $v \in M_{1,n}(\mathbb{C}) \otimes 1_P M 1_Q$ *satisfying* $av = v\theta(a)$ *for all* $a \in P$.

(2) *There exist projections* $p \in P$, $q \in Q$, *a normal unital* $\star$*-homomorphism* $\theta : pPp \to qQq$ *and a non-zero partial isometry* $v \in pMq$ *satisfying* $av = v\theta(a)$ *for all* $a \in pPp$.

(3) *There exists no sequence of unitaries* $(u_n)_{n\in\mathbb{N}} \in \mathcal{U}(P)$ *such that for all* $x, y \in 1_P M 1_Q$, $\lim_n \|E_Q(x^\star u_n y)\|_2 = 0$.

We write $P \prec_M Q$ if the statements in Theorem 2.11 holds. We will use the notion of intertwining-by-bimodules in combination with the following lemma. A detailed version of the proof of the lemma can be found in [Fal09, Lemma 8.10].

For every von Neumann subalgebra $P \subset M$, we consider the *normalizer* $\mathcal{N}_M(P) = \{u \in \mathcal{U}(M) \mid uPu^\star = P\}$. When $\mathcal{N}_M(P)'' = M$, we say that $P \subset M$ is *regular*. When moreover $P \subset M$ is maximal abelian, we call $P \subset M$ a Cartan subalgebra. Finally, when P and M are II_1 factors, we say that $P \subset M$ is an irreducible subfactor if $P' \cap M = \mathbb{C}1$.

Lemma 2.12 ([IPP05, Lemma 8.4])**.** *Let* M *be a* II_1 *factor and* $P_0, P_1 \subset M$ *regular, irreducible subfactors. Assume that the groups* $\frac{\mathcal{N}_M(P_i)}{\mathcal{U}(P_i)}$ *do not admit nontrivial finite normal subgroups. Suppose that*

$$P_0 \prec P_1 \quad \text{and} \quad P_1 \prec P_0.$$

Then there exists a unitary $u \in \mathcal{U}(M)$ *such that* $uP_0u^\star = P_1$.

Using the main results of [PV11, PV12], the assumptions $P_0 \prec P_1$ and $P_1 \prec P_0$ in the previous lemma are automatically satisfied when the algebras P_i are amenable and the groups $\frac{\mathcal{N}_M(P_i)}{\mathcal{U}(P_i)}$ are hyperbolic (or satisfy other rank 1 type conditions). The same holds for many other classes of groups and, as we will see, they all belong to one natural family of groups that we introduce in Section 2.3.3 below.

We will also need the following elementary lemma, which gives a precise meaning to an intertwining $A \prec B$ between *abelian* subalgebras. This result can be proved in the same way as the well-known theorem by Popa [Pop02, Theorem A.1] stating that for Cartan subalgebras $A, B \subset N$, an intertwining $A \prec B$ is equivalent to the existence of a unitary $u \in \mathcal{U}(N)$ such that $uAu^\star = B$.

Lemma 2.13. *Let* N *be a finite von Neumann algebra. If* $A, B \subset N$ *are abelian subalgebras with* $A = (A' \cap N)' \cap N$, $B = (B' \cap N)' \cap N$ *and* $A \prec_N B$, *then there* *exists a nonzero partial isometry $w \in N$ with $ww^\star \in A' \cap N$, $w^\star w \in B' \cap N$ and $Aw \subset wB$.*

Proof. Let $A, B \subset N$ as in the statement of the lemma. Denote by P and Q the relative commutants of A and B, i.e. $P = A' \cap N$ and $Q = B' \cap N$. Since $A \prec_N B$, we also have that $Q \prec_N P$, see [Vae07, Lemma 3.5]. Hence, we find projections $p \in P, q \in Q$, a partial isometry $v \in N$ with $vv^\star \leq q$, $v^\star v \leq p$ and a normal unital $\star$-homomorphism $\theta : qQq \to pPp$ satisfying

$$xv = v\theta(x) \quad \text{for all} \quad x \in qQq.$$

Note that $vv^\star \in (qQq)' \cap qNq = qB$. Write now $D = \theta(qQq)' \cap pNp$ and $f = v^\star v \in D$, then by spatiality, we have

$$fDf = (\theta(qQq)f)' \cap fNf = (v^\star Qv)' \cap fNf = v^\star(qB)v.$$

Hence f is an abelian projection of D. Note that $pA \subset D$ is an abelian subalgebra. Take now $C \subset D$ a maximal abelian subalgebra satisfying $pA \subset C$, and observe that necessarily $C \subset pPp$. Since D is a finite von Neumann algebra, we find a partial isometry $v_1 \in D$ such that $v_1 v_1^\star = f$ and $v_1^\star D v_1 \subset C$, see e.g. [Vae06, Lemma C.2]. Put $w = vv_1$, then we still have

$$xw = w\theta(x) \quad \text{for all} \quad x \in qQq,$$

since v_1 commutes with $\theta(qQq)$, and $ww^\star = vfv^\star = vv^\star \in qB \subset qQq$. Moreover, now we obtain $w^\star w = v_1^\star f v_1 \in C \subset pPp$. We get that $w^\star(qQq)w \subset pPp$, hence $ww^\star Qww^\star \subset wPw^\star$. Taking the relative commutants in $ww^\star N ww^\star$, we obtain that $wAw^\star \subset ww^\star B$, hence $w^\star$ is the desired partial isometry. □

2.3.2 On inclusions of von Neumann algebras

In this section, we provide several known results on inclusions of tracial von Neumann algebras, that will turn out to be useful for the definition of the class $\mathcal{C}$ in Section 2.3.3, and for proving basic stability properties of this class.

Let (M, τ) be a tracial von Neumann algebra, and let $P \subset M$ be a subalgebra. The relative size of the inclusion $P \subset M$ can be measured by the so-called *Jones index*, which measures $L^2(M)$ as a P-module. For the sake of self-containment, we also include a definition of left modules and of the Jones index below. We refer to e.g. [JS97] for a more elaborate introduction on modules over a von Neumann algebra and on subfactors.

A *left module* over a von Neumann algebra M is a Hilbert space H together with a $\star$-homomorphism $\pi : M \to B(H)$. We will make use of the following proposition, stating that every module over a von Neumann algebra can be written in a special form.

Proposition 2.14 ([JS97, Theorem 2.2.2]). *Let (P, τ) be a tracial von Neumann algebra, and write $P^{op} = P' \cap B(L^2(P)) = J_\tau P J_\tau$. For any separable left P-module H, there exists a projection $p \in \ell^\infty(\mathbb{N}) \mathbin{\overline{\otimes}} P^{op}$ and an isometric isomorphism $U : H \to p\big(\ell^2(\mathbb{N}) \otimes L^2(P)\big)$ such that $U(x \cdot \xi) = (\mathrm{id} \otimes x)U(\xi)$ for all $x \in P, \xi \in H$.*

Using the previous proposition, we now can define the *dimension* with respect to τ of H as a P-module as $\dim_P H = \sum_{i=0}^\infty \tau(E_{\mathcal{Z}(\mathcal{P})}(p_i))$, where $p = (p_i)_{i\in\mathbb{N}} \in \ell^\infty(\mathbb{N}) \mathbin{\overline{\otimes}} P^{\mathrm{op}}$ is any projection such that $H \cong p(\ell^2(\mathbb{N}) \otimes L^2(P))$ and $E_{\mathcal{Z}(P)} : P^{\mathrm{op}} \to \mathcal{Z}(P)$ is the conditional expectation onto the center. Let now $P \subset M$ be an inclusion of tracial von Neumann algebras. The *Jones index* of the inclusion $P \subset M$ with respect to the trace τ is then defined as the $\dim_P L^2(M)$, the dimension of $L^2(M)$ when considered as a left P-module, and the inclusion $P \subset M$ is said to have *finite Jones index* if $\dim_P L^2(M) < \infty$. We note that if $\Lambda < \Gamma$ is an inclusion of discrete groups, the Jones index of the inclusion $L(\Lambda) \subset L(\Gamma)$ is exactly given by the index $[\Gamma : \Lambda]$ as groups.

Since for every inclusion of tracial von Neumann algebras $P \subset M$, the left P-module $L^2(P)$ naturally embeds into $L^2(M)$, we also have the following reformulation. Identify $L^2(P)$ as subspace of $L^2(M)$, and denote by $e_P : L^2(M) \to L^2(P)$ the orthogonal projection. By the previous proposition, we then find countably many partial isometries $v_i \in P' \cap B(L^2(M))$ such that $v_i^\star v_i \leq e_P$, such that $v_i v_i^\star$ is an orthogonal family of projections, and such that $L^2(M) = \sum_{i\in\mathbb{N}} v_i L^2(P)$. In this way, $B(L^2(M)) \cap P'$ is a von Neumann algebra carrying a semifinite tracial weight given by $\hat{\tau}(x) = \sum_{i=1}^\infty \tau(E_{\mathcal{Z}(P)}(v_i^\star x v_i))$, and the Jones index of the inclusion $P \subset M$ with respect to the trace τ is equal to $\hat{\tau}(1)$.

An inclusion $P \subset M$ is said to be *essentially of finite index* if there exist projections $p \in P' \cap M$ that lie arbitrarily close to 1 such that $Pp \subset pMp$ has finite Jones index. Note that being essentially of finite index is independent of the choice of faithful trace τ, as it is equivalent with the existence of projections $p \in P' \cap M$ arbitrarily close to 1 such that $\sum_{i=1}^\infty E_{\mathcal{Z}(P)}(v_i^\star p v_i) < \infty$[3]. For later use, we also make the following observation, which we formulate as a folklore lemma.

[3] To make sense of the series $\sum_{i=1}^\infty E_{\mathcal{Z}(P)}(v_i^\star p v_i)$, one can identify $\mathcal{Z}(P) = L^\infty([0,1])$. Any faithful trace τ then corresponds with a probability measure equivalent to the Lebesgue measure.

Lemma 2.15 (Folklore). *Let $A \subset B$ be an inclusion of abelian von Neumann algebras. Then the inclusion $A \subset B$ is essentially of finite index if and only if there exists projections $p_k \in B$ such that $\sum_k p_k = 1$ and $Ap_k = Bp_k$ for all k.*

Proof. It is clear that the existence of projections $p_k \in B$ with $\sum_k p_k = 1$ and $Ap_k = Bp_k$ implies that the inclusion $A \subset B$ is essentially of finite index. For the converse implication, it is enough to show that for every finite index inclusion $A \subset B$, there exists countably many projections $p_k \in B$ with $\sum_k p_k = 1$ and $Ap_k = Bp_k$. Let $A \subset B$ now be such a finite index inclusion. Identify $L^2(A)$ as a subspace of $L^2(B)$, and denote by $e_A : L^2(B) \to L^2(A)$ the orthogonal projection. By Proposition 2.14, we find partial isometries $v_i \in A' \cap B(L^2(B))$ such that $v_i^\star v_i \leq e_A$, $v_i v_i^\star$ are pairwise orthogonal, and $L^2(B) = \sum_{i\in\mathbb{N}} v_i L^2(A)$. Note that $B \subset B(L^2(B))$ is a maximal abelian subalgebra, and that

$$v_i^\star v_i(B(L^2(B)) \cap A')v_i^\star v_i = v_i^\star v_i(B(L^2(A)) \cap A')v_i^\star v_i = v_i^\star v_i A,$$

hence the $v_i v_i^\star$'s form a family of orthogonal abelian projections, thus $B(L^2(B)) \cap A'$ is a type I von Neumann algebra. Denote by $D \subset B(L^2(B)) \cap A'$ a maximal abelian subalgebra containing the projections $v_i v_i^\star$, and let $u \in B(L^2(B)) \cap A'$ denote a unitary such that $uDu^\star = B$, which exists since $A' \cap B(L^2(B))$ is a finite type I von Neumann algebra [Vae06, Lemma C.2]. Put $p_i = uv_i v_i^\star u^\star$, then $p_i \in B$, $\sum_i p_i = 1$ and we still have that $L^2(B) = \sum_i uv_i L^2(A)$. In particular, $p_i L^2(B) = uv_i L^2(A)$, and we have for all $x \in Bp_i$ that $(uv_i)^\star x(uv_i) \in B(L^2(A)) \cap A' = A$. In particular, $x \in (uv_i)A(uv_i)^\star = (uv_i)(uv_i)^\star A = p_i A$, and we conclude that $Bp_i = Ap_i$. $\square$

In the proof of Lemma 2.21 below, we need the following results from [JP81], on normalizers of corners of subalgebras. For the convenience of the reader, we also provide a proof.

Proposition 2.16 ([JP81, Remark 2.4]). *Let (M, τ) be a tracial von Neumann algebra and $A \subset M$ a von Neumann subalgebra. Let $\alpha \in \mathrm{Aut}(A)$ and $x \in M$ such that $xa = \alpha(a)x$ for all $a \in A$. Denote by z, z' the right and left support projections of x, and assume that $z, z' \in \mathcal{Z}(A)$. Let $e \in \mathcal{Z}(A)$ be a projection that dominates z and z'. Then, $x \in \mathcal{N}_{eMe}(Ae)''$.*

Proof. Remark that $x^\star x \in (Ae)' \cap eMe \subset \mathcal{N}_{eMe}(Ae)''$, hence replacing x by its polar part, we may assume that x is a partial isometry with right support z and left support z' satisfying $va = \alpha(a)v$ for all $a \in A$. Denoting by β the restriction of α to Az, it follows that $\beta : Az \to Az'$ is an isomorphism satisfying $\beta(a) = vav^\star$ for all $a \in Az$. In particular, β is trace-preserving.

Put now for every $n \geq 1$, $e_n = v^n(v^\star)^n$ and $e'_n = (v^\star)^n v^n$. Note that

$$e_n = \alpha^{n-1}(z')\alpha^{n-2}(z') \cdots \alpha(z')z' \quad \text{and}$$

$$e'_n = \alpha^{-1+n}(z)\alpha^{-2+n}(z) \cdots \alpha^{-1}(z)z$$

are decreasing sequences of projections in $\mathcal{Z}(Ae)$. Put now for every $n \geq 1$, $z_n = (e_n - e_{n+1})(1-z)$ and $z'_n = \beta^{-n}(z_n) = (v^\star)^n z_n v^n$. One computes that $z'_n = (e'_n - e'_{n+1})(1-z')$. We claim that $\sum_n z_n = 1 - z$. To prove the claim, put $f = \bigwedge_{n\geq 1} e_n(1-z)$, we will show that $f = 0$. For every $n \geq 1$, we have that

$$\beta^{-n}(f) \leq \beta^{-n}(e_n(1-z)) = e'_n - e'_{n+1}.$$

Hence the projections $\beta^{-n}(f)$ are all orthogonal, and as β preserves the finite trace τ, and it follows that indeed $f = 0$. Similarly, for $f' = \bigwedge_{n\geq 1} e'_n(1-z')$ we get that $\beta^n(f)$ is an orthogonal family, and hence $\sum_n z'_n = 1 - z'$. It follows that

$$u = e(1-z)(1-z') + v + \sum_{n\geq 1} (v^\star)^n z_n$$

is a unitary operator in eMe satisfying $uau^\star = \theta(a)$ for all $a \in Ae$, with $\theta \in \mathrm{Aut}(Ae)$ the automorphism defined by

$$\theta(a) = \begin{cases} a & \text{if } a \in Ae(1-z)(1-z'), \\ \beta(a) & \text{if } a \in Az, \\ \beta^{-n}(a) & \text{if } a \in Az_n, n \geq 1. \end{cases}$$

So, $u \in \mathcal{N}_{eMe}(Ae)$ and since $v = uz$, we conclude that that $v \in \mathcal{N}_{eMe}(Ae)''$. □

As a direct consequence of the previous corollary, we have the following result on normalizers of corners of subalgebras. Also compare with [HU15, Proposition 2.3].

Corollary 2.17 ([JP81, Proof of 2.1]). *Let (M, τ) be a tracial von Neumann algebra and $A \subset M$ a von Neumann subalgebra. If $z \in \mathcal{Z}(A)$ is a projection, then $\mathcal{N}_{zMz}(Az)'' = z\mathcal{N}_M(A)''z$.*

Proof. If $u \in \mathcal{N}_{zMz}(Az)$, then $u+(1-z)$ belongs to $\mathcal{N}_M(A)$ and $z(u+(1-z))z = u$, and it follows that $\mathcal{N}_{zMz}(Az)'' \subset z\mathcal{N}_M(A)''z$.

For the converse inclusion, assume that $u \in \mathcal{N}_M(A)$ and let $\alpha \in \mathrm{Aut}(A)$ denote the automorphism given by $\alpha = \mathrm{Ad}\, u$. Put $x = zuz$ and note that $xa = \alpha(a)x$ for all $a \in A$, and that the left and right support projections of x are given by $z\alpha(z)$ and $z\alpha^{-1}(z)$ respectively. By Proposition 2.16, it follows that $x \in \mathcal{N}_{zMz}(Az)''$, showing the converse inclusion. □

2.3.3 A class $\mathcal{C}$ of groups

We call $A \subset M$ a *virtual core subalgebra* if $A' \cap M = \mathcal{Z}(A)$ and if the inclusion $\mathcal{N}_M(A)'' \subset M$ is essentially of finite index.

Definition 2.18. We say that a countably infinite group Γ belongs to class $\mathcal{C}$ if for every trace-preserving cocycle action $\Gamma \curvearrowright (B, \tau)$ and every amenable, virtual core subalgebra $A \subset p(B \rtimes \Gamma)p$, we have that $A \prec B$. ▲

Note that all groups in the class $\mathcal{C}$ are nonamenable: this follows by taking $B = \mathbb{C}1$ and $A = L(\Gamma)$. Similarly, groups in the class $\mathcal{C}$ do not admit infinite amenable normal subgroups: if $H \lhd \Gamma$ is a normal subgroup and $\Gamma/H \curvearrowright (X, \mu)$ is a free probability measure preserving action, then $L^\infty(X) \rtimes H \subset L^\infty(X) \rtimes \Gamma$ is a virtual core subalgebra, and $L^\infty(X) \rtimes H \prec L^\infty(X)$ implies that H is finite, as otherwise any sequence $(h_n)_{n\in\mathbb{N}} \in H$ with $h_n \to \infty$ would violate Theorem 2.11 (3) (with $u_n = \lambda(h_n)$).

Remark 2.19. We have several motivations to introduce this class $\mathcal{C}$ of groups. Our definition is first of all motivated by the concept of $\mathcal{C}_s$-rigidity introduced in [PV11, Definition 1.4]. A group Γ is called $\mathcal{C}_s$-rigid if the following property holds. For every free ergodic probability measure preserving action of Γ on (X, μ) with crossed product $M = L^\infty(X) \rtimes \Gamma$ and for any maximal abelian von Neumann subalgebra $A \subset M$ whose normalizer is a finite index subfactor of M, we have that A is unitarily conjugate with $L^\infty(X)$. By [Pop02, Theorem A.1], every group in the class $\mathcal{C}$ is also $\mathcal{C}_s$-rigid. In particular, all groups in the class $\mathcal{C}$ are Cartan-rigid (or $\mathcal{C}$-rigid), meaning that for all crossed product II_1 factors $L^\infty(X) \rtimes \Gamma$ by free ergodic probability measure preserving actions, $L^\infty(X)$ is the unique Cartan subalgebra up to unitary conjugacy.

Also, by Lemma 2.12, if Γ and Λ are icc groups in the class $\mathcal{C}$ and if we have a stable isomorphism of crossed products $\pi : R \rtimes \Gamma \to (R \rtimes \Lambda)^t$ given by outer actions of Γ, Λ on the hyperfinite II_1 factor R, then $\pi(R)$ is unitarily conjugate to R^t and $\Gamma \cong \Lambda$. This is the result that we need in our thesis.

So, class $\mathcal{C}$ unifies Cartan-rigidity and unique crossed product decompositions for outer actions. Moreover, as we prove below, the class $\mathcal{C}$ satisfies several stability properties, making it a natural class of groups to consider. In particular, we prove in Proposition 2.22 that class $\mathcal{C}$ is stable under extensions. This stability result is motivated by [CIK13], where the structural theorems of [PV11, PV12] for crossed products with free groups or hyperbolic groups are used to show that also extensions of hyperbolic groups by hyperbolic groups are Cartan-rigid.

Finally note that over the last few years, large classes of groups have been shown to belong to class $\mathcal{C}$. Assume that $\Gamma \curvearrowright (B, \tau)$ is a cocycle action. Write

$M = B \rtimes \Gamma$. To 'remove' the cocycle, consider the dual coaction $\Delta : M \to M \mathbin{\overline{\otimes}} L(\Gamma)$ defined by $\Delta(bu_g) = bu_g \otimes u_g$ for all $b \in B$, $g \in \Gamma$. Whenever $A \subset pMp$ is an amenable, virtual core subalgebra, we write $\tilde{p} = \Delta(p)$ and we can apply the following results to the inclusion $\Delta(A) \subset \tilde{p}(M \mathbin{\overline{\otimes}} L(\Gamma))\tilde{p}$. Note that, in order to prove that $A \prec B$, it suffices to show that $\Delta(A) \prec M \otimes 1$: if $\Delta(A) \not\prec M \otimes 1$, then there exists a sequence of unitaries $(u_n)_{n\in\mathbb{N}} \in \mathcal{U}(A)$ such that $\|E_{M\otimes 1}(x\Delta(u_n)y)\|_2 \to 0$ for all $x, y \in M \mathbin{\overline{\otimes}} L(\Gamma)$, and since for $x, y \in \Delta(M)$,

$$\|E_{M\otimes 1}(x\Delta(u_n)y)\|_2 = \|E_{M\otimes 1}(E_{\Delta(M)}(x\Delta(u_n)y))\|_2 = \|E_{\Delta(B)}(x\Delta(u_n)y)\|_2,$$

it follows that also $\Delta(A) \not\prec_{\Delta(M)} \Delta(B)$, thus $A \not\prec_M B$.

- By [PV11, Theorem 3.1, Lemma 4.1 and Theorem 7.1], every weakly amenable group Γ with $\beta_1^{(2)}(\Gamma) > 0$ belongs to the class $\mathcal{C}$.

  Let Γ be such a group, let $\Gamma \curvearrowright (B, \tau)$ be a cocycle action and put $M = B \rtimes \Gamma$. Take any amenable virtual core subalgebra $A \subset pMp$, and take a nonzero projection $q \in A' \cap pMp = \mathcal{Z}(A)$ such that $\mathcal{N}_{qMq}(Aq)'' = q\mathcal{N}_{pMp}(A)''q \subset qMq$ has finite index. The proof of [PV11, Lemma 4.1] shows that [PV11, Theorem 3.1] still holds for the *cocycle* action $\Gamma \curvearrowright (B, \tau)$, i.e. for any 1-cocycle $c : \Gamma \to K_\mathbb{R}$ into an orthogonal representation $\eta : \Gamma \to \mathcal{O}(K_\mathbb{R})$, at least one of the following statements holds. We use the notations of section 3 in [PV11], in particular $P = \mathcal{N}_{qMq}(Aq)$.

  - The qMq-M-bimodule ${}_{qMq}(q\mathcal{K}^\eta)_M$ is left P-amenable; or
  - there exist $t, \delta > 0$ such that $\|\widetilde{\psi}_t(\Delta(a))\|_2 \geq \delta$ for all $a \in \mathcal{U}(Aq)$.

  The first part of the proof of [PV11, Theorem 7.1] shows that the first statement implies that η is an amenable representation. Since $\beta_1^{(2)}(\Gamma) > 0$, we know that Γ is nonamenable and that Γ admits an unbounded 1-cocycle c into a nonamenable mixing representation $\eta : \Gamma \to \mathcal{O}(K_\mathbb{R})$. Fixing this representation, we thus get $t, \delta > 0$ such that $\|\widetilde{\psi}_t(a)\|_2 \geq \delta$ for all $a \in \mathcal{U}(\Delta(Aq))$.

  If now $\Delta(A) \not\prec M \otimes 1$, then also $\Delta(Aq) \not\prec M \otimes 1$, and it follows as in the last paragraph on page 34 of [PV11], applied to the trivial crossed product $M \mathbin{\overline{\otimes}} L(\Gamma)$, that there exists a projection $z \in \mathcal{Z}(\Delta(M))$ and a $t > 0$ such that

  $$\|\widetilde{\psi}_t(x)z\|_2 \geq \|z\|_2/2 \quad \text{for all } x \in \mathcal{U}(\Delta(M)).$$

  Since $c : \Gamma \to K_\mathbb{R}$ is unbounded, we can take a sequence $g_n \in \Gamma$ such that $\|c(g_n)\| \to \infty$. But then $\|\widetilde{\psi}_t(\Delta(u_{g_n}))\|_2 = \|\widetilde{\psi}_t(u_{g_n} \otimes u_{g_n})\|_2 \to 0$ as $n \to \infty$, hence also $\|\widetilde{\psi}_t(\Delta(u_{g_n})z)\|_2 \to 0$, contradiction.

- By [PV12, Theorem 3.1], all weakly amenable, nonamenable, bi-exact groups (and in particular, all nonelementary hyperbolic groups) belong to the class $\mathcal{C}$.

  To show this statement, we need the notion of *relative amenability.* If (M, τ) is a tracial von Neumann algebra and $Q \subset M$ is a subalgebra, we say that $P \subset pMp$ is *amenable relative to Q inside M* if there exists a state φ on $p\langle M, e_Q\rangle p = p\big(B(L^2(M)) \cap (Q^{\mathrm{op}})'\big)p$ such that $\varphi|_{pMp} = \tau_{pMp}$ and $\varphi(xy) = \varphi(yx)$ for all $x \in P$, $y \in p\langle M, e_Q\rangle p$.

  Let now Γ be such a group and assume that $M = B \rtimes \Gamma$ and $A \subset pMp$ is an amenable virtual core subalgebra. Take a projection $q \in A' \subset pMp$ such that $\mathcal{N}_{qMq}(Aq)'' \subset qMq$ has finite index, as above, and put $\tilde{q} = \Delta(q)$. If $\Delta(A) \not\prec M \otimes 1$, [PV12, Theorem 3.1] implies that $\mathcal{N}_{\tilde{q}(M \overline{\otimes} L(\Gamma))\tilde{q}}(\Delta(Aq))''$ is amenable relative to $M \otimes 1$. The final part of the proof of [PV12, Proposition 3.2] then shows that $\mathcal{N}_{qMq}(Aq)''$ is amenable relative to B, and [PV12, Lemma 2.2] then implies that M is amenable relative to B.

  Note that in the explicit construction of the crossed product $B \rtimes \Gamma$ as in Section 2.2, $1 \otimes \ell^\infty(\Gamma) \subset B(L^2(B) \otimes \ell^2(\Gamma)) \cap (\pi(B)^{\mathrm{op}})'$, and that with $(g \cdot f)(h) = f(g^{-1}h)$ the action of Γ on $\ell^\infty(\Gamma)$, we have that $1 \otimes (g \cdot f) = \lambda(g)(1 \otimes f)\lambda(g)^\star$ for $f \in \ell^\infty(\Gamma)$, $g \in \Gamma$. Let φ be a state on $\langle M, e_B\rangle$ implementing the relative amenability of M to B inside M, then $m(f) = \varphi(1 \otimes f)$ defines a left invariant state on $\ell^\infty(\Gamma)$. Hence Γ is amenable, contradiction.

- By [Ioa12, Theorem 1.6] (see also [Vae13, Theorem A]), every free product $\Gamma = \Gamma_1 \star \Gamma_2$ with $|\Gamma_1| \geq 2$ and $|\Gamma_2| \geq 3$ belongs to the class $\mathcal{C}$.

  Let $\Gamma = \Gamma_1 * \Gamma_2$ be such a group, and assume that $M = B \rtimes \Gamma$ has an amenable virtual core subalgebra $A \subset pMp$. Again take a projection $q \in A' \subset pMp$ such that $\mathcal{N}_{qMq}(Aq)'' \subset qMq$ has finite index. Note that we can write $M = (B \rtimes \Gamma_1) *_B (B \rtimes \Gamma_2)$. By [Ioa12, Theorem 1.6], it follows that at least one of the following statements is true.

  - $Aq \prec_M B$;
  - there is an $i \in \{1, 2\}$ such that $\mathcal{N}_{qMq}(Aq)'' \prec_M B \rtimes \Gamma_i$; or
  - $\mathcal{N}_{qMq}(Aq)''$ is amenable relative to B inside M.

  If the first statement holds, we are done, since it implies that $A \prec_M B$. Since $\mathcal{N}_{qMq}(Aq)'' \subset qMq$ has finite index, the second statement implies that $M \prec_M B \rtimes \Gamma_i$, see [Vae07, Lemma 3.9]. But this is impossible, as any sequence $(g_n)_{n\in\mathbb{N}} \in \Gamma_1 * \Gamma_2$ such that the word length of g_n goes to infinity, violates Theorem 2.11 (3) (with $u_n = \lambda(g_n)$).

Finally, the third statement would as above imply that M is amenable relative to B inside M, as $\mathcal{N}_{qMq}(Aq)'' \subset qMq$ has finite index. As above, this would imply that Γ is amenable, contradiction. $\diamondsuit$

The proof of the following lemma is essentially contained in the proof of [Pop02, Theorem A.2]. Recall that a projection $p \in M$ is called *abelian* if pMp is abelian, and that a von Neumann algebra M is of type I if all nonzero projections $p \in M$ contain a nonzero abelian subprojection $q \leq p$.

Lemma 2.20. *Let (M, τ) be a tracial von Neumann algebra and $Q \subset N \subset M$ von Neumann subalgebras. Assume that $q \in Q' \cap M$ is a nonzero projection such that $q(Q' \cap M)q = \mathcal{Z}(Q)q$. Then there exists a nonzero projection $p \in Q' \cap N$ such that $p(Q' \cap N)p = \mathcal{Z}(Q)p$ and such that p is smaller than the support projection of $E_N(q)$.*

Proof. Denote by e the support projection of $E_N(q)$. Note that $q \leq e$, as $\tau(q - qeq) = \tau(q) - \tau(E_N(q)e) = 0$, and that $e \in Q' \cap N$. Replacing $Q \subset N \subset M$ by $Qe \subset eNe \subset eMe$, we may assume that the support projection of $E_N(q)$ equals 1.

Denote by $z \in \mathcal{Z}(Q' \cap M)$ the central support of $q \in Q' \cap M$. Since $q(Q' \cap M)q = \mathcal{Z}(Q)q$, the projection $q \in Q' \cap M$ is abelian and $(Q' \cap M)z$ is of type I. We claim that the center of $(Q' \cap M)z$ is equal to $\mathcal{Z}(Q)z$. Note that $\mathcal{Z}(Q)z \subset \mathcal{Z}((Q' \cap M)z)$. To prove the other inclusion, take $x \in \mathcal{Z}((Q' \cap M)z)$, and note that $xq \in q(Q' \cap M)q = \mathcal{Z}(Q)q$, hence we find $y \in \mathcal{Z}(Q)$ such that $xq = yq$. Then $xE_{\mathcal{Z}((Q' \cap M)z)}(q) = yE_{\mathcal{Z}((Q' \cap M)z)}(q)$, and since $E_{\mathcal{Z}((Q' \cap M)z)}(q)$ has support projection z, it follows that $x = y$, proving the claim.

In particular, the inclusion $\mathcal{Z}(Q)z \subset (Q' \cap M)z$ is essentially of finite index. A fortiori, the inclusion $\mathcal{Z}(Q)z \subset (Q' \cap N)z$ is essentially of finite index. This inclusion is isomorphic with the inclusion $\mathcal{Z}(Q)f \subset (Q' \cap N)f$, where f denotes the support projection of $E_N(z)$. Since $\mathcal{Z}(Q)f$ lies in the center of $(Q' \cap N)f$, it follows that $(Q' \cap N)f$ is of type I with $\mathcal{Z}(Q)f$ being an essentially finite index subalgebra of its center, as $(Q' \cap N)f \subset B(L^2((Q' \cap N)f)) \cap (\mathcal{Z}(Q)f)'$ and the latter is of type I. Because $(Q' \cap N)f$ is of type I, we can choose a projection $p_1 \in (Q' \cap N)f$ such that $p_1(Q' \cap N)p_1 = \mathcal{Z}(Q' \cap N)p_1$. Since $\mathcal{Z}(Q)p_1 \subset \mathcal{Z}(Q' \cap N)p_1$ is an inclusion of abelian von Neumann algebras of essentially finite index, we finally find a nonzero projection $p \in \mathcal{Z}(Q' \cap N)p_1$ such that $\mathcal{Z}(Q)p = \mathcal{Z}(Q' \cap N)p = p(Q' \cap N)p$. $\square$

In the following lemma, we prove that virtual cores behave well with respect to Popa's intertwining-by-bimodules. This will be crucial in proving that class $\mathcal{C}$ is stable under extensions. The proof is almost identical to the proof of [CIK13, Proposition 3.6]. We denote $P^n = M_n(\mathbb{C}) \otimes P$.

Lemma 2.21. *Let (M,τ) be a tracial von Neumann algebra. Let $A \subset eMe$ and $N \subset M$ be von Neumann subalgebras. Assume that $A \subset eMe$ is a virtual core subalgebra and that $A \prec N$. Then we can choose nonzero projections $p \in N^n$, $z \in \mathcal{Z}(A)$, an injective normal $\star$-homomorphism $\theta : Az \to pN^n p$ and a partial isometry $v \in (M_{1,n}(\mathbb{C}) \otimes M)p$ satisfying*

- *$vv^\star = z$ and $E_{pN^n p}(v^\star v)$ has support projection p.*
- *$av = v\theta(a)$ for all $a \in Az$.*
- *$\theta(Az)$ is a virtual core subalgebra of $pN^n p$.*

Proof. Since $A \prec N$, we can choose a projection $p_1 \in N^n$, a normal $\star$-homomorphism $\theta : A \to p_1 N^n p_1$ and a nonzero partial isometry $w \in e(M_{1,n}(\mathbb{C}) \otimes M)p_1$ satisfying $aw = w\theta(a)$ for all $a \in A$. Write $z_1 = ww^\star$ and note that $z_1 \in A' \cap eMe = \mathcal{Z}(A)$. Note that the restriction of θ to Az_1 is injective.

Denote $q_1 = w^\star w$. We may assume that the support projection of $E_{p_1 N^n p_1}(q_1)$ equals p_1. Since $A' \cap eMe = \mathcal{Z}(A)$, the inclusions $\theta(Az_1) \subset p_1 N^n p_1 \subset p_1 M^n p_1$ and the projection q_1 satisfy the assumptions of Lemma 2.20. So by Lemma 2.20, we can choose a nonzero projection $p \in p_1 N^n p_1 \cap \theta(Az_1)'$ such that

$$(\theta(Az_1)p)' \cap pN^n p = \mathcal{Z}(\theta(Az_1))p \; .$$

We define v as the polar part of wp and replace θ by $\theta(\cdot)p$. Put $z = vv^\star$, then we still have that $z \in \mathcal{Z}(A)$ and that the restriction of θ to Az is injective. We now also have that $\theta(Az)' \cap pN^n p = \mathcal{Z}(\theta(Az))$. Write $q = v^\star v$. By construction, the support projection of $E_{pN^n p}(q)$ equals p.

Define $P = \mathcal{N}_{pN^n p}(\theta(Az))''$. Let $u \in \mathcal{N}_{zMz}(Az)$ and define $x := E_{pN^n p}(v^\star uv)$. We prove that x belongs to P. Define the automorphism $\alpha \in \mathrm{Aut}(\theta(Az))$ given by $\alpha = \theta \circ (\mathrm{Ad}\, u) \circ \theta^{-1}$. By construction, $xa = \alpha(a)x$ for all $a \in \theta(Az)$. Since the relative commutant of $\theta(Az)$ in $pN^n p$ equals $\mathcal{Z}(\theta(Az))$, it follows from Proposition 2.16 that indeed $x \in P$.

Define $P_0 = \mathcal{N}_{zMz}(Az)''$. Because $z \in \mathcal{Z}(A)$, it follows from Corollary 2.17 that $P_0 = z\mathcal{N}_{eMe}(A)''z$. Since $A \subset eMe$ is a virtual core subalgebra, we conclude that $P_0 \subset zMz$ is essentially of finite index. Thus $v^\star P_0 v \subset qM^n q$ is essentially of finite index. Define $P_1 \subset pN^n p$ as the von Neumann subalgebra generated by $E_{pN^n p}(v^\star P_0 v)$. By [CIK13, Lemma 2.3], the inclusion $P_1 \subset pN^n p$ is essentially of finite index. In the previous paragraph, we have proved that $P_1 \subset P$. So a fortiori, $P \subset pN^n p$ is essentially of finite index. So we have proved that $\theta(Az) \subset pN^n p$ is a virtual core subalgebra. $\square$

We can now prove that the class $\mathcal{C}$ is stable under extensions.

Proposition 2.22. *The class $\mathcal{C}$ is stable under extensions.*

Proof. Assume that $\Gamma_1 \lhd \Gamma$ with $\Gamma/\Gamma_1 \cong \Gamma_2$ and that $\Gamma_1, \Gamma_2 \in \mathcal{C}$. We have to prove that $\Gamma \in \mathcal{C}$. Choose a trace-preserving cocycle action $\Gamma \curvearrowright (B, \tau)$. Write $M = B \rtimes \Gamma$ and let $A \subset eMe$ be an amenable virtual core subalgebra. We have to prove that $A \prec B$. Write $N = B \rtimes \Gamma_1$. We can then view $M = N \rtimes \Gamma_2$ for some cocycle action $\Gamma_2 \curvearrowright N$, as follows: for every $s \in \Gamma_2$, choose a representative $r(s) \in \Gamma$ such that $s = r(s)\Gamma_1$. Then $\Gamma_2 \curvearrowright^{\alpha, v} N$ with $\alpha_s(x) = \lambda(r(s))x\lambda(r(s))^\star$ and $v_{s,t} = \lambda(r(s)r(t)r(st)^{-1}) \in L(\Gamma_1)$ defines a cocycle action such that $N \rtimes \Gamma_2 = M$. Because Γ_2 belongs to $\mathcal{C}$, we have that $A \prec N$. Choose projections $z \in \mathcal{Z}(A)$, $p \in N^n$, a partial isometry $v \in (M_{1,n}(\mathbb{C}) \otimes M)p$ and a $\star$-homomorphism $\theta : Az \to pN^np$ satisfying the conclusions of Lemma 2.21.

Since $N^n = (M_n(\mathbb{C}) \otimes B) \rtimes \Gamma_1$ for some cocycle action of Γ_1 on $M_n(\mathbb{C}) \otimes B$, and because $\Gamma_1 \in \mathcal{C}$, it follows that $\theta(Az) \prec M_n(\mathbb{C}) \otimes B$ inside N^n. Since the support of $E_{pN^np}(v^\star v)$ equals p, this intertwining can be combined with the intertwining given by v and we obtain the conclusion that $A \prec B$ inside M. $\square$

Recall that two groups Γ, Λ are called *commensurable* if they admit isomorphic finite index subgroups.

Proposition 2.23. *The class $\mathcal{C}$ is stable under commensurability.*

Proof. Let $\Lambda < \Gamma$ be a finite index subgroup. We have to prove that $\Lambda \in \mathcal{C}$ if and only if $\Gamma \in \mathcal{C}$. Write $n = [\Gamma : \Lambda]$ and $H = \Gamma/\Lambda$, and choose for every $h \in H$ a representative $r(h) \in \Gamma$ such that $h = r(h)\Lambda$.

First assume that $\Gamma \in \mathcal{C}$. Let $\Lambda \curvearrowright (B, \tau)$ be a trace-preserving cocycle action. Write $M = B \rtimes \Lambda$ and let $A \subset eMe$ be an amenable virtual core subalgebra. Define $B_1 = \ell^\infty(H) \otimes B$ and consider the induced cocycle action $\Gamma \curvearrowright B_1$ given by $g \cdot (f \otimes x) = (g \cdot f) \otimes (g \cdot x)$ for $g \in \Gamma$, $f \in \ell^\infty(H)$ and $x \in B$, where $(g \cdot f)(h) = f(g^{-1}h)$. Writing as in the previous proposition $B_1 \rtimes \Gamma$ as the crossed product of the cocycle action $H \curvearrowright^{\alpha, v} B_1 \rtimes \Lambda$ given by $\alpha_s = \operatorname{Ad} \lambda(r(s)), v_{s,t} = \lambda(r(s)r(t)r(st)^{-1})$ for $s, t \in H$, we get

$$B_1 \rtimes \Gamma \cong (B_1 \rtimes \Lambda) \rtimes H \cong \big(\ell^\infty(H) \otimes (B \rtimes \Lambda)\big) \rtimes H.$$

Note that the latter crossed product acts naturally on $\ell^2(H) \otimes \ell^2(H) \otimes L^2(B)$, as it is generated by the operators

$$\lambda(h) : \delta_s \otimes \delta_t \otimes \xi \mapsto \delta_{hs} \otimes \delta_t \otimes \xi, \quad s, t, h \in H, \xi \in L^2(B \rtimes \Lambda),$$

$$\pi(f \otimes 1) : \delta_s \otimes \delta_t \otimes \xi \mapsto f(st)\delta_s \otimes \delta_t \otimes \xi, \quad f \in \ell^\infty(\Gamma/\Lambda),$$

$$\pi(1 \otimes x) : \delta_s \otimes \delta_t \otimes \xi \mapsto \delta_s \otimes \delta_t \otimes \alpha_{s^{-1}}(x)\xi, \quad x \in B \rtimes \Lambda.$$

Via the spatial isomorphism $\delta_s \otimes \delta_t \otimes \xi \mapsto \delta_{st} \otimes \delta_t \otimes \lambda(r(s))\xi$, where $\lambda(r(s))\hat{x} = \lambda(r(s))x\lambda(r(s))\hat{1}$ for $x \in B \rtimes \Lambda$, one sees that $\big(\ell^\infty(H) \otimes (B \rtimes \Lambda)\big) \rtimes H$ is isomorphic to $M_n(\mathbb{C}) \otimes 1 \otimes (B \rtimes \Lambda)$. In particular we can identify $B_1 \rtimes \Gamma \cong M_n(\mathbb{C}) \otimes (B \rtimes \Lambda)$. In this way, we can view $M_n(\mathbb{C}) \otimes A$ as an amenable virtual core subalgebra of a corner of $B_1 \rtimes \Gamma$. Since $\Gamma \in \mathcal{C}$, it follows that $M_n(\mathbb{C}) \otimes A \prec B_1$ and thus, $A \prec B$.

Conversely assume that $\Lambda \in \mathcal{C}$. Let $\Gamma \curvearrowright (B, \tau)$ be a trace-preserving cocycle action. Write $M = B \rtimes \Gamma$ and let $A \subset eMe$ be an amenable virtual core subalgebra. Write $N = B \rtimes \Lambda$. Since $\Lambda < \Gamma$ has finite index, we have $A \prec N$. We apply Lemma 2.21 and find the virtual core subalgebra $\theta(Az) \subset pN^n p$. Since $\Lambda \in \mathcal{C}$, it follows that $\theta(Az) \prec B$, but then also $A \prec B$, as in the final paragraph of the proof of Proposition 2.22. $\square$

2.4 Structural properties of infinite tensor products

In this section, we give a detailed definition of the infinite tensor product of von Neumann algebras equipped with normal faithful states, and we study the properties of the resulting von Neumann algebra, provided all tensor factors are the same.

Definition 2.24. Let I be a countably infinite set, and let (M_i, φ_i) be a von Neumann algebra equipped with a normal faithful state, for every $i \in I$. The infinite tensor product von Neumann algebra of the (M_i, φ_i)'s is defined as follows. Denote by H the infinite tensor product of the Hilbert spaces $L^2(M_i, \varphi_i)$ with unit vectors $\eta_i(1)$, i.e. H is the L^2-completion of the set of all finite linear combinations of simple tensor vectors $\otimes_{i \in I}\xi_i$, where $\xi_i \in L^2(M_i, \varphi_i)$ for all $i \in I$ and $\xi_i = \eta_i(1)$ for all but finitely many $i \in I$, with respect to the inproduct given by $\langle \otimes_{i\in I}\xi_i, \otimes_{i\in I}\zeta_i\rangle = \prod_{i \in I}\langle \xi_i, \zeta_i\rangle_{\varphi_i}$. For every $k \in I$, we have the representation π_k of M_k on H given by $\pi_k(x)\big(\otimes_{i\in I}\xi_i\big) = \otimes_{i\in I}\zeta_i$, with $\zeta_i = \xi_i$ if $i \neq k$ and $\zeta_k = \pi_{\varphi_k}(x)\xi_k$. Then the infinite tensor product of the von Neumann algebras M_i with respect to the states φ_i is the subalgebra of $B(H)$

defined by

$$\overline{\bigotimes}_{i\in I} M_i = \{\pi_i(x) \mid i \in I, x \in M_i\}''. \qquad \blacktriangle$$

Fix now a von Neumann algebra (P, ϕ) with a normal faithful state ϕ. Whenever I is a countable set, we write P^I for the tensor product of P indexed by I with respect to ϕ. The canonical product state on P^I will be denoted by ϕ^I. We will now study the structure of the infinite tensor product (P^I, ϕ^I). The first result we get, is a type classification for these infinite tensor products. We show in particular that such tensor products never yield a factor of type III_0. This result is probably well known, but we could not find a reference in the literature.

Lemma 2.25. *Let (P, ϕ) be a nontrivial factor equipped with a normal faithful state, and let I be a countable infinite set. The factor $(P, \phi)^I$ is of type*

II_1 *if ϕ is tracial,*

$\mathrm{III}_\lambda, \lambda \in (0,1)$ *if ϕ is nontracial and periodic with period $\frac{2\pi}{|\log \lambda|}$, and*

III_1 *if ϕ is not periodic.*

Moreover, denoting by $P^I \rtimes \mathbb{R}$ the crossed product with the modular action of ϕ^I, we have that $\mathcal{Z}(P^I \rtimes \mathbb{R}) = L(G)$, where $G < \mathbb{R}$ is the subgroup given by $G = \{t \in \mathbb{R} \mid \sigma_t^\phi = \mathrm{id}\}$.

Proof. Put $\varphi = \phi^I$. If ϕ is tracial, then clearly φ is a trace on $(P, \phi)^I$, hence $(P, \phi)^I$ is a type II_1 factor and clearly $\mathcal{Z}(P^I \rtimes \mathbb{R}) = L(\mathbb{R})$. Assume now that ϕ is not tracial. Then $(P, \phi)^I$ cannot be semifinite and thus is a type III factor, see e.g. [Tak03b, Theorem XIV.1.14].

Identify $I = \mathbb{N}$ and let $N = P^{\mathbb{N}} \rtimes \mathbb{R}$ denote the crossed product with respect to the modular group σ^{ϕ^I}. Also consider the crossed product $(P \mathbin{\overline{\otimes}} P^{\mathbb{N}}) \rtimes \mathbb{R}$ for the diagonal action of $\mathbb{R}$ on P and $P^{\mathbb{N}}$. For every $n \in \mathbb{N}$, let α_n denote the $\star$-isomorphism $N \to (P \mathbin{\overline{\otimes}} P^{\mathbb{N}}) \rtimes \mathbb{R}$ defined by

$$\alpha_n\big(\pi_\sigma(\otimes_k x_k)\big) = \pi_\sigma\big(x_n \otimes \big(x_0 \otimes x_1 \otimes \cdots \otimes x_{n-1} \otimes x_{n+1} \otimes \cdots\big)\big), \quad \text{for } x_k \in P,$$

$$\alpha_n(\lambda(t)) = \lambda(t), \quad \text{for } t \in \mathbb{R}.$$

Let $\iota : N \to (1 \otimes P^{\mathbb{N}}) \rtimes \mathbb{R} \subset (P \mathbin{\overline{\otimes}} P^{\mathbb{N}}) \rtimes \mathbb{R}$ be the canonical embedding, and remark that for all $x \in N$, $\alpha_n(x) \to \iota(x)$ $\star$-strongly. Then for every element $x \in \mathcal{Z}(N)$, we have that $\iota(x) \in \mathcal{Z}((P \mathbin{\overline{\otimes}} P^{\mathbb{N}}) \rtimes \mathbb{R})$, and in particular $[\iota(x), \pi_\sigma(a \otimes 1)] = 0$ for all $a \in P$.

Consider the explicit realization of the crossed product $(P \mathbin{\overline{\otimes}} P^{\mathbb{N}}) \rtimes \mathbb{R}$ on the Hilbert space $L^2\big(\mathbb{R}, L^2(P) \otimes L^2(P^{\mathbb{N}})\big)$ as given in Section 2.1.2. For any state ω on P, consider the completely positive map $\overline{\omega} : (P \mathbin{\overline{\otimes}} P^{\mathbb{N}}) \rtimes \mathbb{R} \to B(L^2(\mathbb{R}, L^2(P^{\mathbb{N}})))$, by applying ω to the first copy of $\pi_\sigma(P)$ in the crossed product:

$$\{\big(\overline{\omega}\pi_\sigma(x \otimes y)\big)\xi\}(s) = \omega(\sigma^{\phi}_{-s}(x))\sigma^{\phi^{\mathbb{N}}}_{-s}(y)\xi(s), \qquad \text{for } x \in P,\ y \in P^{\mathbb{N}}, s, t \in \mathbb{R},$$

$$\{\big(\overline{\omega}\lambda(t)\big)\xi\}(s) = \xi(t^{-1}s), \qquad\qquad \xi \in L^2(\mathbb{R}, L^2(P^{\mathbb{N}})).$$

Note that for all $x \in N$, $\overline{\omega}\big(\iota(x)\big) = x$, hence $\iota(N)$ belongs to the multiplicative domain of $\overline{\omega}$. Take now $x \in \mathcal{Z}(N)$ and $a \in P$, then $[\iota(x), \pi_\sigma(a \otimes 1)] = 0$ implies that the elements $\overline{\omega}\big(\iota(x)\big) = x$ and $\overline{\omega}(\pi_\sigma(a \otimes 1))$ in $B(L^2(P^{\mathbb{N}}) \otimes L^2(\mathbb{R}))$ commute. But note that

$$\{\overline{\omega}(\pi_\sigma(a \otimes 1))\xi\}(s) = \omega(\sigma^{\phi}_{-s}(a))\xi(s) \quad \text{for } \xi \in L^2(\mathbb{R}, L^2(P^{\mathbb{N}})).$$

This means that $\mathcal{Z}(N) \subset (1 \otimes D)' \cap B(L^2(N))$, where $D \subset L^\infty(\mathbb{R})$ is the von Neumann subalgebra generated by the functions $t \mapsto \omega(\sigma^{\phi}_{-t}(a))$ for $\omega \in P_\star, a \in P$. On the other hand, we have by Lemma 2.8 that $\mathcal{Z}(N) \subset (L\mathbb{R})' \cap N = (P^{\mathbb{N}})_\varphi \mathbin{\overline{\otimes}} L\mathbb{R}$. Combining both facts, we obtain that $\mathcal{Z}(N) \subset (P^{\mathbb{N}})_\varphi \mathbin{\overline{\otimes}} L\mathbb{R} \cap (1 \otimes D)' = (P^{\mathbb{N}})_\varphi \mathbin{\overline{\otimes}} (L\mathbb{R} \cap D')$.

We now make the following case distinction. **Case 1:** The state ϕ is not periodic. In this case, the functions $t \mapsto \omega(\sigma^{\phi}_{-t}(a))$ for $\omega \in P_\star, a \in P$ separate points of $\mathbb{R}$, and hence we have that $D = L^\infty(\mathbb{R})$. In particular, $L\mathbb{R} \cap D' = \mathbb{C}$, meaning that $\mathcal{Z}(N) \subset (P^{\mathbb{N}})_\varphi$. As $P^{\mathbb{N}}$ is a factor, we conclude that N is also a factor, and hence $P^{\mathbb{N}}$ is of type III_1.

Case 2: The state ϕ is periodic with period $T > 0$. Then D consists of all bounded functions $f \in L^\infty(\mathbb{R})$ that are T-periodic, i.e. $f(s) = f(s + T)$ for all $s \in \mathbb{R}$. In particular, D contains the function $f : t \mapsto \exp(\frac{2\pi \mathbf{i} t}{T})$. Using the Fourier transform $L\mathbb{R} \cong L^\infty(\hat{\mathbb{R}})$, it is now easy to see that $L\mathbb{R} \cap D' \subset L\mathbb{R} \cap \{M_f\}' = L(T\mathbb{Z})$, and hence $\mathcal{Z}(N) \subset (P^{\mathbb{N}})_\varphi \mathbin{\overline{\otimes}} L(T\mathbb{Z})$. Using the Fourier decomposition of elements in $P^{\mathbb{N}} \rtimes T\mathbb{Z}$, we now can conclude that $\mathcal{Z}(N) = L(T\mathbb{Z})$, and hence $P^{\mathbb{N}}$ is a factor of type III_λ with $\lambda = e^{-\frac{2\pi}{T}}$. $\square$

For $\mu \in \mathbb{R}^+_0$, we denote by $P_{\phi,\mu}$ the eigenvectors of Δ_ϕ for μ, i.e.

$$P_{\phi,\mu} = \{x \in P \mid \Delta_\phi \hat{x} = \mu \hat{x}\} = \{x \in P \mid \sigma^t_\phi(x) = \mu^{\mathbf{i}t} x\}.$$

Recall that a state ϕ is called *almost periodic* if Δ_ϕ is diagonalizable. In general, we define $P_{\phi,\mathrm{ap}} \subset P$ as the subalgebra spanned by the eigenvectors of Δ_ϕ, i.e.

$$P_{\phi,\mathrm{ap}} = \big(\operatorname{span} \bigcup_{\mu \in \mathbb{R}^+_0} P_{\phi,\mu}\big)''.$$

The notation $P_{\phi,\mathrm{ap}}$ stands for the *almost periodic part* of P, and this name is justified since $P_{\phi,\mathrm{ap}}$ is the maximal subalgebra $Q \subset P$ with *ϕ-preserving conditional expectation*, such that the restriction of ϕ to Q is almost periodic. Here, a subalgebra $Q \subset P$ is said to be with ϕ-preserving conditional expectation, if there exists a conditional expectation $E : P \to Q$ satisfying $\phi \circ E = \phi$. Note that a ϕ-preserving conditional expectation is automatically normal, see Theorem 2.6.

In Lemma 2.29 below, we will show that if (P, ϕ) is a nontrivial factor equipped with a normal faithful state and I is an infinite set, then the centralizer $(P^I)_{\phi^I}$ of the infinite tensor product is a factor if and only if $P_{\phi,\mathrm{ap}}$ is a factor. A first step in proving this result, is handling the case where ϕ is almost periodic and hence $P_{\phi,\mathrm{ap}} = P$. The following lemma is a generalization of [CS74, Theorem 4.4] and [Pop01, Proposition 2.2.2]. For the convenience of the reader, we provide a complete proof.

Lemma 2.26. *Let (P, ϕ) be a factor, equipped with a normal faithful almost periodic state ϕ. Let I be a countably infinite set. Then $((P^I)_{\phi^I})' \cap P^I = \mathbb{C}1$.*

Proof. We can take $I = \mathbb{N}$. Denote $(M, \varphi) = (P, \phi)^{\mathbb{N}}$. Let $\pi_n : P \to M$ be the canonical embedding of P at the n-th position. Let $\Gamma \subset \mathbb{R}_0^+$ be the subgroup generated by the point spectrum of Δ_φ.

For any $n \in \mathbb{N}$, consider the state-preserving $\star$-isomorphism $\alpha_n : M \to P \otimes M$ defined by

$$\alpha_n(\otimes_k x_k) = x_n \otimes \big(x_0 \otimes x_1 \otimes \cdots \otimes x_{n-1} \otimes x_{n+1} \otimes \cdots\big).$$

Remark that for every $x \in M$, $\alpha_n(x) \to 1 \otimes x$ $\star$-strongly. Let now $x \in M \cap M'_\varphi$. Since $\alpha_n(x) \in P \otimes M \cap (P \otimes M)'_{\phi\otimes\varphi}$ for all $n \in \mathbb{N}$, we have $1 \otimes x \in P \otimes M \cap (P \otimes M)'_{\phi\otimes\varphi}$. Take $y \in \bigcup_{\gamma\in\Gamma} P_{\phi,\gamma}$. We have that $y^\star \otimes \pi_n(y) \in (P \otimes M)_{\phi\otimes\varphi}$. This means that x commutes with $\pi_n(y)$. Since the linear span of $\bigcup_{\gamma\in\gamma} P_{\phi,\gamma}$ is $\star$-strongly dense in P, we conclude that $x \in M \cap M' = \mathbb{C}1$. ☐

To generalize the result from Lemma 2.26 to the case where $P_{\phi,\mathrm{ap}} \subsetneq P$ is a strict subfactor, we need the following elementary lemmas.

Lemma 2.27. *Let (M, ψ) be a von Neumann algebra equipped with a normal faithful state. Then $\overline{M_{\psi,\mu}}^{\|\cdot\|_\psi} = \{\xi \in L^2(M, \psi) \mid \Delta_\psi \xi = \mu\xi\}$ for every $\mu \in \mathbb{R}_0^+$.*

Proof. Fix $\mu \in \mathbb{R}_0^+$ and take $T > 0$ such that $\mu^{\mathrm{i}T} = 1$. Denote by E_T the ψ-preversing conditional expectation $M \to M^{\sigma_T^\psi}$ onto the fixed points under σ_T^ψ. We claim that for every $x \in M$, we have that $E_T(x)\hat{1} = P_{\ker \Delta_\psi^{\mathrm{i}T} - \mathrm{id}}(\hat{x})$. To prove the claim, let $x \in M$, and consider L to be the set of weak limit points of $W := \mathrm{conv}\{\sigma^{\psi}_{nT}(x) \mid n \in \mathbb{Z}\}$, i.e. of all convex combinations of the $\sigma^{\psi}_{nT}(x)$, for $n \in \mathbb{Z}$. As the set W is bounded, L is nonempty, and by construction, for all $y \in L$, also $\sigma^{\psi}_{T}(y) \in L$. Further remark that L is convex and $\|\cdot\|_{\psi}$-closed. Taking $\tilde{x} \in L$ with minimal $\|\cdot\|_{\psi}$-norm, we get that $\sigma^{\psi}_{T}(\tilde{x}) = \tilde{x}$, and in particular $P_{\ker \Delta_{\psi}^{\mathrm{i}T} - \mathrm{id}}(\tilde{x}\hat{1}) = \tilde{x}\hat{1}$ and $E_T(\tilde{x}) = \tilde{x}$.

Take now a sequence $x_n \in W$ such that $x_n \to \tilde{x}$ weakly, and observe that $P_{\ker \Delta_{\psi}^{\mathrm{i}T} - \mathrm{id}}(x_n\hat{1}) = P_{\ker \Delta_{\psi}^{\mathrm{i}T} - \mathrm{id}}(x\hat{1})$ and $E_T(x_n) = E_T(x)$ for all $n \in \mathbb{N}$. On the other hand, since $P_{\ker \Delta_{\psi}^{\mathrm{i}T} - \mathrm{id}}$ and E_T are weakly continuous, we also get that

$$\lim_{n\to\infty} P_{\ker \Delta_{\psi}^{\mathrm{i}T} - \mathrm{id}}(x_n\hat{1}) = P_{\ker \Delta_{\psi}^{\mathrm{i}T} - \mathrm{id}}(\tilde{x}\hat{1}) = \tilde{x}\hat{1},$$

$$\lim_{n\to\infty} E_T(x_n) = E_T(\tilde{x}) = \tilde{x}.$$

This shows that $P_{\ker \Delta_{\psi}^{\mathrm{i}T} - \mathrm{id}}(x\hat{1}) = E_T(x)\hat{1} \in M^{\sigma^{\psi}_{T}}$, which proves the claim.

In particular, it follows that the projection P_μ of $L^2(M)$ onto $\{\xi \in L^2(M) \mid \Delta_\psi \xi = \mu\xi\}$ satisfies $P_\mu(\hat{x}) \in M_{\psi,\mu}\hat{1}$ for all $x \in M$, since

$$\begin{aligned} P_\mu(\hat{x}) &= \frac{1}{T}\int_0^T \mu^{-\mathrm{i}t}\Delta_\psi^{\mathrm{i}t}(P_{\ker \Delta_{\psi}^{\mathrm{i}T} - \mathrm{id}}(\hat{x}))dt \\ &= \frac{1}{T}\int_0^T \mu^{-\mathrm{i}t}\sigma^{\psi}_{t}(E_T(x))\,\hat{1}dt \in M_{\psi,\mu}\hat{1}. \end{aligned}$$

Using that $P_\mu(\hat{M})$ is dense in $\{\xi \in L^2(M) \mid \Delta_\psi\xi = \mu\xi\}$, this proves the lemma. $\square$

Lemma 2.28. *Let (M,ψ), (N,φ) be factors equipped with normal faithful states. For every $\mu \in \mathbb{R}_0^+$, it holds that*

$$(M \mathbin{\overline{\otimes}} N)_{\psi\otimes\varphi,\mu} \subset \bigoplus_{t\in\mathbb{R}_0^+} \overline{M_{\psi,\mu t^{-1}} \otimes N_{\varphi,t}}^{\|\cdot\|_{\psi\otimes\varphi}}, \tag{2.4}$$

as subsets of $L^2(M\mathbin{\overline{\otimes}}N, \psi\otimes\varphi)$. In particular, $(M\mathbin{\overline{\otimes}}N)_{\psi\otimes\varphi,ap} = M_{\psi,ap}\mathbin{\overline{\otimes}}N_{\varphi,ap}$.

Proof. Let (M,ψ) and (N,φ) be factors with normal faithful states, and fix $\mu \in \mathbb{R}_0^+$. In this proof, we will write $L^2(M)$ and $L^2(N)$ for $L^2(M,\psi)$ and $L^2(N,\varphi)$ respectively. Assume that $x \in (M \mathbin{\overline{\otimes}} N)_{\psi\overline{\otimes}\varphi,\mu}$, and consider $\hat{x} \in L^2(M) \otimes L^2(N)$. Note that $(\Delta_\psi \otimes \Delta_\varphi)\hat{x} = \mu\hat{x}$, hence identifying $L^2(M) \otimes$

$L^2(N) \cong \mathcal{HS}(\overline{L^2M}, L^2N)$ by $\xi \otimes \eta \mapsto \langle \cdot, \overline{\xi} \rangle \eta$, $\hat{x}$ corresponds to an operator $T \in \mathcal{HS}(\overline{L^2M}, L^2N)$ satisfying $\Delta_\varphi T \overline{\Delta}_\psi = \mu T$. In particular, $T^\star T \in \mathcal{TC}(\overline{L^2M})$ commutes with $\overline{\Delta}_\psi$, and therefore, we can find an orthogonal family of unit vectors $\xi_k \in L^2M$ and $c_k, t_k \in \mathbb{R}_0^+$ such that $T^\star T = \sum_{k\in\mathbb{N}} c_k \langle \cdot, \overline{\xi}_k \rangle \overline{\xi}_k$ and $\overline{\Delta}_\psi \overline{\xi}_k = t_k \overline{\xi}_k$. Putting $\eta_k = T\overline{\xi}_k$, we get that $(\eta_k)_k$ is an orthogonal family such that $T = \sum_{k\in\mathbb{N}} \langle \cdot, \overline{\xi}_k \rangle \eta_k$ and $\Delta_\varphi \eta_k = \Delta_\varphi T(t_k^{-1} \Delta_\psi \overline{\xi}_k) = \mu t_k^{-1} \eta_k$. Note that $\hat{x} = \sum_{k\in\mathbb{N}} \xi_k \otimes \eta_k$, and hence (2.4) follows now from Lemma 2.27.

It follows immediately from (2.4) that $(M \mathbin{\overline{\otimes}} N)_{\psi\otimes\varphi,\mu} \subset L^2(M_{\psi,\mathrm{ap}} \mathbin{\overline{\otimes}} N_{\varphi,\mathrm{ap}}, \psi \otimes \varphi)$, and hence for every $x \in (M \mathbin{\overline{\otimes}} N)_{\psi\otimes\varphi,\mu}$ we get $E_{M_{\psi,\mathrm{ap}} \overline{\otimes} N_{\varphi,\mathrm{ap}}}(x) = x$. This demonstrates that indeed $(M \mathbin{\overline{\otimes}} N)_{\psi\otimes\varphi,\mathrm{ap}} = M_{\psi,\mathrm{ap}} \mathbin{\overline{\otimes}} N_{\varphi,\mathrm{ap}}$. □

Lemma 2.29. *Let (P, ϕ) be a nontrivial factor equipped with a normal faithful state, and let I be a countable infinite set. Then it holds that $(P^I)_{\phi^I} = (P^I_{\phi,ap})_{\phi^I}$. In particular, the following two statements are equivalent:*

(i) $(P^I)_{\phi^I}$ is a factor.

(ii) $P_{\phi,ap}$ is a factor.

Proof. Let P, I be as in the statement, and put $Q = P_{\phi,\mathrm{ap}}$ and $\varphi = \phi^I$. The inclusion $(Q^I)_\varphi \subset (P^I)_\varphi$ being obvious, take $x \in (P^I)_\varphi$; we will show that $x \in Q^I$. Note that for all finite subsets $\mathcal{F} \subset I$, $E_{P^\mathcal{F}}(x) \in (P^\mathcal{F})_\varphi$, and by Lemma 2.28, we have that

$$E_{P^\mathcal{F}}(x) \in (P^\mathcal{F})_\varphi \subset (P^\mathcal{F})_{\varphi,\mathrm{ap}} = Q^\mathcal{F} \subset Q^I.$$

As x is a limit point of $\{E_{P^\mathcal{F}}(x) \mid \mathcal{F} \subset I \text{ finite}\}$ in the strong topology, it follows that also $x \in Q^I$.

The implication (ii) $\Rightarrow$ (i) follows now directly from Lemma 2.26. For the reverse implication, note that $\mathcal{Z}(P_{\phi,\mathrm{ap}}) \subset P_\phi$. Thus, if $x \in \mathcal{Z}(P_{\phi,\mathrm{ap}})$ is a nontrivial element, then $x \otimes 1 \otimes 1 \otimes \cdots$ belongs to $(P^I)_\varphi$, and commutes with all elements of $(P_{\phi,\mathrm{ap}})^I$. In particular, $(P^I)_\varphi = (P^I_{\phi,\mathrm{ap}})_\varphi$ is not a factor. □

2.5 Noncommutative Bernoulli actions

Let (P, ϕ) be a von Neumann algebra with a normal faithful state ϕ. Whenever I is a countable set, we write P^I for the tensor product of P indexed by I with respect to ϕ. The canonical product state on P^I will be denoted by ϕ^I.

Consider now a countable group Λ that acts on I, and let Λ act on P^I by the (generalized) Bernoulli action

$$\rho(s)\big(\otimes_{k\in I}\, a_k\big) = \otimes_{k\in I} a_{s^{-1}\cdot k}, \qquad\qquad \text{for } s\in\Lambda, a_h\in P.$$

The von Neumann algebra (P,ϕ) is called the *base algebra* for the Bernoulli action, and the crossed product $P^I \rtimes \Lambda$ is called the *Bernoulli crossed product*.

The following probably well-known lemma provides a criterion for the generalized Bernoulli action to be properly outer. Recall that an automorphism ρ on a von Neumann algebra M is properly outer if there exists no nonzero element $v\in M$ satisfying $vx=\rho(x)v$ for all $x\in M$. We say that P has a direct summand Q, if there exists a central projection $z\in\mathcal{Z}(P)$ such that $zP=Q$.

Lemma 2.30. *Let (P,ϕ) be a nontrivial von Neumann algebra equipped with a normal faithful state ϕ. Let I be a countable infinite set. Denote by P^I the infinite tensor product with respect to ϕ, and by $\pi_k : P\to P^I$ the embedding at position k. Let $\alpha : I\to I$ be any nontrivial permutation, and let $\rho\in \mathrm{Aut}(P^I,\phi^I)$ denote the induced automorphism given by*

$$\rho(\pi_k(x)) = \pi_{\alpha(k)}(x), \quad \textit{for } x\in P, k\in I.$$

Then ρ is properly outer, unless P admits a type I *factor as a direct summand and $\{k\in I \mid \alpha(k)\neq k\}$ is finite.*

More generally, we have the following: if $\theta\in\mathrm{Aut}(P,\phi)$ is a state-preserving automorphism with diagonal product $\theta^I\in\mathrm{Aut}(P^I,\phi^I)$, and if $v\in P^I$ such that $v\theta^I(x)=\rho(x)v$ for all $x\in P^I$, then $v=0$, unless P admits a type I *factor as a direct summand, $\{k\in I\mid\alpha(k)\neq k\}$ is finite and $\theta=\mathrm{id}|_P$.*

Note that when $\{k\in I\mid \alpha(k)\neq k\}$ is finite and P is a finite von Neumann algebra, the lemma is folklore and can be proved as follows. Assume that there exists a nonzero element v as in the statement of the lemma. Then, fixing $k\in I$ such that $\alpha(k)\neq k$, identifying $P^I = P^k\,\overline{\otimes}\,P^{\alpha(k)}\,\overline{\otimes}\,P^{I-\{k,\alpha(k)\}}$ and applying $\mathrm{id}\otimes\mathrm{id}\otimes\omega$ for a well-choosen $\omega\in(P^{I-\{k,\alpha(k)\}})_\star$, we find a nonzero element $w\in P\,\overline{\otimes}\,P$ satisfying $w(x\otimes 1)=(1\otimes\theta(x))w$ for all $x\in P$. Fixing a trace τ on P and identifying $L^2(P,\tau)\otimes L^2(P,\tau)\cong\mathcal{HS}(L^2(P,\tau))$ by $\xi\otimes\eta\mapsto\langle\cdot,J_\tau\xi\rangle\eta$, $\hat{w}$ corresponds to an operator $T\in\mathcal{HS}(L^2(P,\tau))$ satisfying

$$Tx=\theta(x)T \quad \text{for all } x\in P.$$

Taking a spectral projection p of the operator $T^\star T$, we find a projection $p\in B(L^2(P,\tau))\cap P'$ such that $pL^2(P,\tau)$ is finite-dimensional. Denoting by $z\in\mathcal{Z}(P)$ the central support of p, we then get that $(B(L^2(P,\tau))\cap P')z=(Pz)^{\mathrm{op}}$ is of type I, and thus also Pz is a type I von Neumann algebra.

To show the lemma for *general* von Neumann algebras P, we will reduce in the proof below to the case where P is a factor, as then the result follows from the theorem in [Sak73] stating that for any von Neumann algebra Q, the flip automorphism on $Q \overline{\otimes} Q$ given by $x \otimes y \mapsto y \otimes x$ is inner if and only if Q is a type I factor.

Proof of Lemma 2.30. We denote $\|x\|_\phi = \sqrt{\phi(x^\star x)}$ for all $x \in P$, and similarly for $x \in P^I$ using $\varphi = \phi^I$. We distinguish two cases. **Case 1.** *The set* $\{k \in I \mid \alpha(k) \neq k\}$ *is finite.* Assume that $v \in P^I$ is a nonzero element satisfying $v\theta^I(x) = \rho(x)v$ for all $x \in P^I$.

We will first show that $\theta = \mathrm{id}|_P$. Suppose that this is not the case, and let $a \in P$ be an element in P such that the left and right multiplication with a on $L^2(P, \phi)$ have norm less than one, and $\theta(a) \neq a$. Put $\epsilon = \frac{1}{4}\|\theta(a) - a\|_\phi \|v\|_\varphi$. Take $\mathcal{F} \subset I$ a finite set large enough such that $\alpha(k) = k$ for all $k \in I - \mathcal{F}$, $\|v - E_{P^\mathcal{F}}(v)\|_\varphi < \epsilon$ and $\|E_{P^\mathcal{F}}(v)\|_\varphi > \frac{1}{2}\|v\|_\varphi$, where $E_{P^\mathcal{F}}$ denotes the canonical φ-preserving conditional expectation $P^I \to P^\mathcal{F}$. Note now that for $k \in I - \mathcal{F}$,

$$\begin{aligned}
\|v\theta^I(\pi_k(a)) - \rho(\pi_k(a))v\|_\varphi &= \|v\pi_k(\theta(a)) - \pi_k(a)v\|_\varphi \\
&\geq \|E_\mathcal{F}(v)\pi_k(\theta(a)) - \pi_k(a)E_\mathcal{F}(v)\|_\varphi - 2\epsilon \\
&= \|E_\mathcal{F}(v)\|_\varphi \|\theta(a) - a\|_\phi - 2\epsilon \\
&> \frac{1}{2}\|v\|_\varphi \|\theta(a) - a\|_\phi - 2\epsilon = 0,
\end{aligned}$$

contradiction. We have now shown that θ is indeed the identity on P.

Let now $\mathcal{F} \subset \{k \in I \mid \alpha(k) \neq k\}$ denote a nontrivial cycle of the permutation α. Identifying $P^I = P^\mathcal{F} \overline{\otimes} P^{I-\mathcal{F}}$ and applying $\mathrm{id} \otimes \omega$ for a well-choosen $\omega \in (P^{I-\mathcal{F}})_\star$, we find a nonzero element $w \in P^\mathcal{F}$ satisfying $wx = \rho(x)w$ for all $x \in P^\mathcal{F}$. We may assume that w is a partial isometry. Fix $k \in \mathcal{F}$ and let z denote the support projection of $\pi_k^{-1}(E_{P^{\{k\}}}(ww^\star))$. Note that z belongs to the center of P. We claim that $z \in \mathcal{Z}(P)$ has a minimal subprojection. To prove the claim, assume that $p \in \mathcal{Z}(P)$ is a projection such that $p \leq z$, and put $q = z - p$. Take $x = \pi_k(p)\pi_{\alpha(k)}(q)$, then $\pi_k(p)\pi_{\alpha(k)}(q)w = w\pi_{\alpha(k)}(p)\pi_{\alpha^2(k)}(q) = \pi_{\alpha(k)}(p)\pi_{\alpha^2(k)}(q)w$, and it follows that $\pi_k(p)\pi_{\alpha(k)}(q)w = 0$ since the left-most and right-most sides of the equation are orthogonal for the inproduct given by φ. If now $\{p_1, \cdots, p_n\}$ are orthogonal projections such that $z = \sum_{i=1}^n p_i$, then it follows that

$$\varphi(ww^\star) = \sum_{i=1}^n \varphi(\pi_k(p_i)\pi_{\alpha(k)}(p_i)ww^\star) + \varphi(\pi_k(p_i)\pi_{\alpha(k)}(z - p_i)ww^\star)$$

$$\leq \sum_{i=1}^{n} \varphi(\pi_k(p_i)\pi_{\alpha(k)}(p_i)) + 0 = \sum_{i=1}^{n} \phi(p_i)^2 \leq \phi(z) \cdot \min_{1\leq i\leq n} \phi(p_i).$$

As $ww^\star$ is nonzero, we see that $z\mathcal{Z}(P)$ cannot be diffuse.

Let now $z_0 \in \mathcal{Z}(P)$ be a minimal projection with $z_0 \leq z$. Note that $\pi_k(z_0)w \neq 0$, and that for all $x \in P^{\mathcal{F}}$, $\pi_k(z_0)wx = \rho(x)\pi_k(z_0)w$, hence by the polar decomposition, we find a partial isometry $w_0 \in P^{\mathcal{F}}$ with $w_0 x = \rho(x)w_0$ for all $x \in P^{\mathcal{F}}$ and $w_0 w_0^\star \leq \pi_k(z_0)$. Note that since $w_0w_0^\star$ and $w_0^\star w_0$ both belong the the center of $P^{\mathcal{F}}$, we have $w_0w_0^\star = w_0^\star w_0$, and since $w_0\pi_{\alpha^{-1}(k)}(z_0) = \pi_k(z_0)w_0 = w_0$, we get that also $w_0w_0^\star = w_0^\star w_0 \leq \pi_{\alpha^{-1}(k)}(z_0)$. Similarly, we obtain that for all $\ell \in \mathcal{F}$, $w_0w_0^\star \leq \pi_\ell(z_0)$. Denoting $\mathcal{F} = \{k_1, \cdots, k_n\}$, we get that $w_0w_0^\star \leq \pi_{k_1}(z_0)\cdots\pi_{k_n}(z_0)$, and as $\pi_{k_1}(z_0)\cdots\pi_{k_n}(z_0) \in \mathcal{Z}(P^{\mathcal{F}})$ is a minimal projection, we get $w_0w_0^\star = \pi_{k_1}(z_0)\cdots\pi_{k_n}(z_0)$. Hence w_0 is a unitary in $(z_0P)^{\mathcal{F}}$ satisfying $w_0x = \rho(x)w_0$ for all $x \in z_0P$.

The result now follows from the well known observation that for any nontrivial permutation σ on n elements and any von Neumann algebra Q, the induced flip automorphism on $Q \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} Q$ given by $x_1 \otimes \cdots \otimes x_n \mapsto x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$ is inner if and only if Q is a type I factor, see e.g. [Sak73, Theorem 5]. Indeed, as a consequence, we get that z_0P must be a type I factor.

Conversely, if P has a type I factor as a direct summand, i.e. if there exists $z \in \mathcal{Z}(P)$ such that zP is a type I factor, and if $\mathcal{F} := \{k \in I \mid \alpha(k) \neq k\}$ is finite, then ρ restricted to the type I factor $(zP)^{\mathcal{F}}$ is inner, and for $u \in (zP)^{\mathcal{F}}$ the unitary implementing ρ, we have $ux = \rho(x)u$ for all $x \in P^I$.

Case 2. *The set $\{k \in I \mid \alpha(k) \neq k\}$ is infinite.* We will show that ρ is properly outer. Fix a nonzero element $a \in P$ with $\phi(a) = 0$ and such that both the left and the right multiplication by a have norm less than 1 on $L^2(P, \phi)$. For every $i \in I$, denote by $\pi_i : P \to P^I$ the embedding in position i. Write I as an increasing union of finite subsets $I_n \subset I$ and denote by E_n the unique state-preserving conditional expectation of P^I onto P^{I_n}.

Assume that $v \in P^I$ is a nonzero element satisfying $v\theta^I(x) = \rho(x)v$ for all $x \in P^I$. Write $\delta = \frac{1}{2}\,\|v\|_\varphi\,\|a\|_\phi$ and note that $\delta > 0$. Take n large enough such that $\|v - E_n(v)\|_\varphi < \delta$ and $\sqrt{2}\|E_n(v)\|_\varphi \geq \|v\|_\varphi$. Since α moves infinitely many elements of I, we can fix an $i \in I$ such that both i and $\alpha(i)$ belong to $I \setminus I_n$ and $i \neq \alpha(i)$. Write $v_n = E_n(v)$.

Note that $\|v\pi_i(\theta(a)) - v_n\pi_i(\theta(a))\|_\varphi \leq \|v - v_n\|_\varphi < \delta$. We also have that $\|\pi_{\alpha(i)}(a)v - \pi_{\alpha(i)}(a)v_n\|_\varphi \leq \|v - v_n\|_\varphi < \delta$. Observe that $\pi_i(\theta(a)) = \theta^I(\pi_i(a))$ and $\pi_{\alpha(i)}(a) = \rho(\pi_i(a))$. Since $v\theta^I(\pi_i(a)) = \rho(\pi_i(a))v$, it follows that $\|v_n\pi_i(\theta(a)) - \pi_{\alpha(i)}(a)v_n\|_\varphi < 2\delta$. Since $\{i\}$, $\{\alpha(i)\}$ and I_n are disjoint subsets of I and since $\phi(a) = 0$, we get that

$$2\delta > \|v_n \pi_i(\theta(a)) - \pi_{\alpha(i)}(a) v_n\|_\varphi = \sqrt{2}\, \|v_n\|_\varphi \, \|a\|_\phi \geq \|v\|_\varphi \, \|a\|_\phi \, .$$

We reached the absurd conclusion that $2\delta > \|v\|_\varphi \, \|a\|_\phi$. □

It is a folklore result that crossed products of properly outer actions on factors are factors, and hence we obtain the following corollary.

Corollary 2.31. *Let (P, ϕ) be a nontrivial von Neumann algebra equipped with a normal faithful state, and assume that $\Lambda \curvearrowright I$ is a faithful action of a countable group on a countable infinite set I. Then the Bernoulli crossed product $P^I \rtimes \Lambda$ is a factor if and only if at least one of the following conditions holds:*

(i) *The orbits of the action $\Lambda \curvearrowright I$ are infinite, or*

(ii) *For every $s \in \Lambda - \{e\}$ with finite conjugacy class, the set $\{k \in I \mid s \cdot k \neq k\}$ is either infinite or intersects an infinite orbit, and P is a factor, or*

(iii) *P is a factor, but not of type* I.

Proof. Denote by $\Lambda_0 < \Lambda$ the subgroup of elements with finite conjugacy classes, and note that we necessarily have that $\mathcal{Z}(P^I \rtimes \Lambda) \subset P^I \rtimes \Lambda_0$, see Lemma 2.9. Put $I_0 \subset I$ the union of all finite orbits. We first claim that the action $\Lambda \curvearrowright L^2(P^I \rtimes \Lambda_0 \ominus P^{I_0} \rtimes \Lambda_0)$ by conjugation with $\lambda(g)$ has no nontrivial fixed points. Let J be the set of all finite nonempty subsets of I that intersect $I - I_0$ nontrivially, and note that we have the orthogonal decomposition

$$L^2(P^I \rtimes \Lambda_0 \ominus P^{I_0} \rtimes \Lambda_0, \phi^I) = \bigoplus_{\mathcal{F} \in J} L^2((P - \mathbb{C}1)^{\mathcal{F}}) \otimes \ell^2(\Lambda_0).$$

Suppose that $x \in L^2(P^I \rtimes \Lambda_0 \ominus P^{I_0} \rtimes \Lambda_0)$ satisfies $s \cdot x = x$ for all $s \in \Lambda$, then we can write $x = \sum_{\mathcal{F} \in J} x_{\mathcal{F}}$ with $x_{\mathcal{F}} \in L^2((P - \mathbb{C}1)^{\mathcal{F}}) \otimes \ell^2(\Lambda_0)$, and since x is invariant under the action of Λ, we get that $\|x_{\mathcal{F}}\|_2 = \|x_{s \cdot \mathcal{F}}\|_2$ for all $s \in \Lambda, \mathcal{F} \in J$. As the action $\Lambda \curvearrowright I - I_0$ has infinite orbits, also $\Lambda \curvearrowright J$ has infinite orbits, and it follows that $x_{\mathcal{F}} = 0$ for all $\mathcal{F} \in J$ and hence $x = 0$, proving the claim.

Assume now that (i) holds. We immediately get that $\mathcal{Z}(P^I \rtimes \Lambda) \subset (P^I \rtimes \Lambda_0) \cap L(\Lambda)' \subset L(\Lambda_0)$, and since the action is faithful, we get $\mathcal{Z}(P^I \rtimes \Lambda) = \mathbb{C}$.

Next, assume that (ii) holds. Since P is a factor, the elements of Λ_0 not acting properly outerly form a subgroup, call it Λ_1. Combining the above claim with Lemma 2.9, we find that $\mathcal{Z}(P^I \rtimes \Lambda) \subset P^{I_0} \rtimes \Lambda_1$. Now take $z \in \mathcal{Z}(P^I \rtimes \Lambda)$, then we can write $z = \sum_{s\in\Lambda_1} x_s\lambda(s)$ for elements $x_s \in P^{I_0}$. Fix now $s \in \Lambda_1 - \{e\}$. Since s does not act properly outerly, the set $\{k \in I \mid s\cdot k \neq k\}$ is finite by the above lemma, and thus we can find some $k \in I - I_0$ such that $s \cdot k \neq k$. Take $y \in P - \mathbb{C}1$ a nontrivial element, and note that $\pi_k(y)$ commutes with x_s. Expressing that $\pi_k(y)$ commutes with z now yields that $x_s\pi_k(y)\lambda(s) = x_s\pi_{s\cdot k}(y)\lambda(s)$, which is only possible if $x_s = 0$. Hence $z \in P^{I_0}$ and $\mathcal{Z}(P^I \rtimes \Lambda) \subset \mathcal{Z}(P^{I_0}) = \mathbb{C}$.

Finally, assume that (iii) holds. Then by the previous lemma, the action $\Lambda \curvearrowright P^I$ is properly outer. Since P^I is a factor, it now follows from Lemma 2.9 that $\mathcal{Z}(P^I \rtimes \Lambda) \subset \mathcal{Z}(P^I) = \mathbb{C}$.

Conversely, assume that conditions (i)–(iii) are all false, we will prove that $P^I \rtimes \Lambda$ is not a factor. If P is not a factor, then take a nontrivial element $x \in \mathcal{Z}(P)$ and put $z = \pi_{k_1}(x)\cdots\pi_{k_n}(x)$, where $\{k_1,\cdots,k_n\} \subset I$ is a finite orbit. One easily checks that $z \in \mathcal{Z}(P^I \rtimes \Lambda)$. Assume now that P is a factor, then by $\neg$(iii) it is of type I. By $\neg$(ii), we find an element $s \in \Lambda_0 - \{e\}$ such that $I_1 := \{k \in I \mid s\cdot k \neq k\}$ is a finite subset of I_0. Let $u \in \mathcal{U}(P^{I_1})$ be a unitary such that $s \cdot x = uxu^\star$ for all $x \in P^I$, and note that $u^\star\lambda(s) \in P^I \rtimes \Lambda \cap (P^I)'$. Denote by $\overline{I_1}$ the union of all orbits of elements in I_1, and consider the action $\mathrm{Sym}(\overline{I_1}) \curvearrowright P^{\overline{I_1}}$ given by $\sigma\cdot\pi_k(y) = \pi_{\sigma(k)}(y)$. Since $\lambda(t)^\star u\lambda(t) \in \{\sigma\cdot u | \sigma \in \mathrm{Sym}(\overline{I_1})\}$ for all $t \in \Lambda$, we see that the set $\{\lambda(t)^\star u^\star\lambda(st) \mid t \in \Lambda\}$ is finite. The sum of all elements in this set is a nontrivial element in $\mathcal{Z}(P^I \rtimes \Lambda)$. □

Remark 2.32. It now easily follows that whenever (P,ϕ) is a nontrivial factor equipped with a normal faithful state, and Λ is an icc group acting faithfully on an infinite countable set I, the resulting Bernoulli crossed product $P^I \rtimes \Lambda$ is a factor of the same type as P^I. Indeed, denoting by $\varphi = \phi^I$ the product state on P^I and by $G \subset \mathbb{R}$ the subgroup given by $G = \{t \in \mathbb{R} \mid \sigma_t^\phi = \mathrm{id}\}$, we have on the one hand that $\mathcal{Z}(P^I \rtimes_{\sigma^\varphi} \mathbb{R}) = L(G) \subset \mathcal{Z}\big((P^I \rtimes_{\sigma^\varphi}\mathbb{R}) \rtimes \Lambda\big)$ by Lemma 2.25. On the other hand $\mathcal{Z}\big((P^I \rtimes_{\sigma^\varphi}\mathbb{R}) \rtimes \Lambda\big) \subset P^I \rtimes_{\sigma^\varphi} \mathbb{R}$ by Lemma 2.9, since Λ is icc and $P^I \rtimes \mathbb{R}$ admits Λ-invariant states, e.g. $\psi(x) = \sum_{n\in\mathbb{N}} \frac{1}{2^{n+1}} \mathrm{Tr}_\varphi(p_nxp_n)$ for projections $(p_n)_{n\in\mathbb{N}} \in L\mathbb{R}$ such that $\mathrm{Tr}_\varphi(p_n) = 1$ and $\sum_{n\in\mathbb{N}} p_n = 1$. Combined we find that $\mathcal{Z}\big((P^I \rtimes \Lambda) \rtimes \mathbb{R}\big) = \mathcal{Z}(P^I \rtimes \mathbb{R})$, showing that $P^I \rtimes \Lambda$ and P^I are indeed factors of the same type.

In Lemma 4.3 below, we will show that $P^I \rtimes \Lambda$ and P^I are also factors of the same type if Λ is not an icc group, but instead P is not of type I, or every element $s \in \Lambda - \{e\}$ having finite conjugacy class moves infinitely many points of I. ♢

Finally, the following lemmas provide a criterion for a generalized Bernoulli crossed product to be full. We first recall the following terminology. An *invariant mean* for an action of a countable group Λ on a set X is a finitely additive Λ-invariant probability measure on all the subsets of X. If $\pi : \Lambda \to \mathcal{U}(H)$ is a unitary representation, a sequence of unit vectors $\xi_n \in H$ is called *almost invariant* if $\|\xi_n - \pi(g)(\xi_n)\| \to 0$ for all $g \in \Lambda$. Note that $\Lambda \curvearrowright X$ admits an invariant mean if and only if the unitary representation $\Lambda \curvearrowright \ell^2(X)$ admits almost invariant unit vectors. A unitary representation $\pi : \Lambda \to \mathcal{U}(H)$ is called *amenable* if there exists a state ω on $B(H)$ such that $\omega(T) = \omega(\pi(g)T\pi(g^{-1}))$ for all $g \in \Lambda, T \in B(H)$, see [Bek89].

Lemma 2.33. *Let (P, ϕ) be a nontrivial von Neumann algebra equipped with a normal faithful state ϕ. Let $\Lambda \curvearrowright I$ be an action of the countable group Λ on the countable set I. The following statements are equivalent.*

(i) *The action $\Lambda \curvearrowright I$ admits an invariant mean.*

(ii) *The action of Λ on the set of all nonempty finite subsets of I admits an invariant mean.*

(iii) *The unitary representation $\Lambda \curvearrowright L^2((P, \phi)^I \ominus \mathbb{C}1)$ admits almost invariant unit vectors.*

(iv) *The unitary representation $\Lambda \curvearrowright L^2((P, \phi)^I \ominus \mathbb{C}1)$ is amenable.*

Proof. We write $H = L^2((P, \phi)^I \ominus \mathbb{C}1)$ and denote by $\rho : \Lambda \curvearrowright H$ the unitary representation given by the Bernoulli action. We denote by J the set of all nonempty finite subsets of I.

(i) $\Rightarrow$ (iii). Fix $a \in P \ominus \mathbb{C}1$ with $\phi(a^\star a) = 1$. Let $\xi_n \in \ell^2(I)$ be a sequence of finitely supported unit vectors satisfying $\lim_n \|g \cdot \xi_n - \xi_n\|_2 = 0$ for all $g \in \Lambda$. For every $i \in I$, denote by $\pi_i : P \to P^I$ the embedding as the i-th tensor factor. Define $\eta_n \in P^I$ given by $\eta_n = \sum_{i \in I} \xi_n(i)\pi_i(a)$. Then η_n is a sequence of almost invariant unit vectors in H.

(iii) $\Rightarrow$ (iv) is trivially true.

(iv) $\Rightarrow$ (ii). For every finite subset $\mathcal{F} \subset I$, denote by $p_{\mathcal{F}}$ the orthogonal projection of H onto the closed linear span of $(P \ominus \mathbb{C}1)^{\mathcal{F}}$. Define the map $\Theta : \ell^\infty(J) \to B(H)$ given by $\Theta(F) = \sum_{\mathcal{F} \in J} F(\mathcal{F}) p_{\mathcal{F}}$. Since $\Theta(g \cdot F) = \rho(g)\Theta(F)\rho(g)^\star$, the composition of an $\operatorname{Ad}\rho(\Lambda)$-invariant mean on $B(H)$ with Θ gives a Λ-invariant mean on J.

(ii) $\Rightarrow$ (i). For every $k \geq 1$, define $V_k \subset I^k$ as the subset of k-tuples that consist of k distinct elements of I. Put $V = \sqcup_{k \geq 1} V_k$. The formula $\theta(i_1, \dots, i_k) = \{i_1, \dots, i_k\}$ defines a finite-to-one Λ-equivariant map $V \to J$. So also the action $\Lambda \curvearrowright V$ admits an invariant mean m. We push forward m along the Λ-equivariant map $V \to I$ given by $(i_1, \dots, i_k) \mapsto i_1$ and find a Λ-invariant mean on I. $\square$

Connes showed in [Con74, Proposition 3.9 (b)] that any Bernoulli crossed product $(P,\phi)^{\mathbb{F}_2} \rtimes \mathbb{F}_2$ of $\mathbb{F}_2$ is a full factor. In fact, any Bernoulli crossed product of a nonamenable group is a full factor, as we see now. For completeness, we provide a complete proof. Recall that a bounded sequence $(x_n)_{n\in\mathbb{N}}$ in a von Neumann algebra M is called *central* if for all $y \in M$, $x_n y - y x_n \to 0$ $\star$-strongly.

Lemma 2.34. *Let (P,ϕ) be a nontrivial von Neumann algebra equipped with a normal faithful state. Let Λ be a countable group acting on the countable set I, such that $\Lambda \curvearrowright I$ has no invariant mean, and consider the Bernoulli action $\Lambda \curvearrowright^{\rho} P^I$. Denote by τ the trace on $L\Lambda$. The following two statements are equivalent.*

(i) *Every $\|\cdot\|_\infty$-bounded central sequence $(y_n)_{n\in\mathbb{N}} \in L\Lambda$ satisfying $\|y_n - E_{L(\operatorname{Stab} i)}(y_n)\|_2 \to 0$ for every $i \in I$, must satisfy $\|y_n - \tau(y_n)1\|_2 \to 0$.*

(ii) *$P^I \rtimes \Lambda$ is a full factor.*

If these conditions hold, then $\tau(P^I \rtimes \Lambda)$ is the weakest topology on $\mathbb{R}$ that makes the function $\sigma^\phi : \mathbb{R} \to \operatorname{Aut}(P) : t \mapsto \sigma^\phi_t$ continuous.

Remark that condition (i) is trivially fulfilled if $I = \Lambda$ and Λ acts on the left, since then $\operatorname{Stab}(\{e\}) = \{e\}$.

Proof. Denote by $\varphi = \phi^I \circ E_{P^I}$ the canonical state on $P^I \rtimes \Lambda$. We write $\|x\|_\varphi = \varphi(x^\star x)^{\frac{1}{2}}$ and $\|x\|^\sharp_\varphi = \varphi(x^\star x + x x^\star)^{\frac{1}{2}}$. Assume that (i) holds. We start by proving the following statement:

$$\text{if } x_n \in P^I \rtimes \Lambda \text{ is a bounded sequence and } t_n \in \mathbb{R} \text{ is any sequence such that } \|x_n \sigma^\varphi_{t_n}(a) - a x_n\|^\sharp_\varphi \to 0 \text{ for all } a \in P^I \rtimes \Lambda, \text{ then } \|x_n - \varphi(x_n)1\|^\sharp_\varphi \to 0. \tag{2.5}$$

Let (x_n) and (t_n) be sequences as in (2.5). We denote by $(u_g)_{g\in\Lambda}$ the canonical unitaries in $P^I \rtimes \Lambda$. We write $H = L^2(P^I \ominus \mathbb{C}1, \varphi)$ and denote by ρ the unitary representation $\Lambda \curvearrowright H$ given by the generalized Bernoulli action. Identifying $L^2(P^I \rtimes \Lambda)$ with the direct sum of $H \otimes \ell^2(\Lambda)$ and $\ell^2(\Lambda)$, the unitary representation $(\operatorname{Ad} u_g)_{g\in\Lambda}$ of Λ on $L^2(P^I \rtimes \Lambda)$ becomes the direct sum of $(\rho_g \otimes \operatorname{Ad} u_g)_{g\in\Lambda}$ and $(\operatorname{Ad} u_g)_{g\in\Lambda}$. Taking $a = u_g$ in (2.5), we can view (x_n) as a sequence of almost $(\operatorname{Ad} u_g)_{g\in\Lambda}$-invariant vectors in $L^2(P^I \rtimes \Lambda)$. By Lemma 2.33, the representation ρ is nonamenable, so that the representation $(\rho_g \otimes \operatorname{Ad} u_g)_{g\in\Lambda}$ does not admit almost invariant unit vectors. We conclude that $\|x_n - E_{L\Lambda}(x_n)\|_\varphi \to 0$. We write $y_n = E_{L\Lambda}(x_n)$. Applying the previous argument to $x_n^\star$, we conclude that $\|x_n - y_n\|^\sharp_\varphi \to 0$. Also, y_n is a central sequence in $L\Lambda$.

Fix a nonzero element $b \in P \ominus \mathbb{C}1$ such that the right multiplication with b is a bounded operator on $L^2(P,\phi)$. Fix $i \in I$ and denote by $\pi_i : P \to P^I$ the embedding as the i-th tensor factor. Taking $a = \pi_i(b)$ in (2.5), we get that

$$\|y_n\, \pi_i(\sigma^\phi_{t_n}(b)) - \pi_i(b)\, y_n\|^\sharp_\varphi \to 0 \; .$$

A direct computation gives that

$$\sqrt{2}\, \|b\|^\sharp_\phi \, \|y_n - E_{L(\mathrm{Stab}\, i)}(y_n)\|_2 \leq \|y_n\, \pi_i(\sigma^\phi_{t_n}(b)) - \pi_i(b)\, y_n\|^\sharp_\varphi \; .$$

It thus follows that $\|y_n - E_{L(\mathrm{Stab}\, i)}(y_n)\|_2 \to 0$ for every $i \in I$. Because (i) holds, we get that $\|y_n - \tau(y_n)1\|_2 \to 0$. This concludes the proof of (2.5).

Taking $t_n = 0$ in (2.5), it follows in particular that $P^I \rtimes \Lambda$ is a full factor. Denote by τ the weakest topology on $\mathbb{R}$ that makes the function $\sigma^\phi : \mathbb{R} \to \mathrm{Aut}(P) : t \mapsto \sigma^\phi_t$ continuous. We prove that $\tau = \tau(P^I \rtimes \Lambda)$. Because $\mathrm{Aut}(P)$ and $\mathrm{Out}(P^I \rtimes \Lambda)$ are Polish groups, it suffices to prove that if $t_n \to 0$ in $\tau(P^I \rtimes \Lambda)$, then also $t_n \to 0$ in τ. In that case, there is a sequence of unitaries $u_n \in P^I \rtimes \Lambda$ such that $(\mathrm{Ad}\, u_n) \circ \sigma^\varphi_{t_n} \to \mathrm{id}$ in $\mathrm{Aut}(P^I \rtimes \Lambda)$. It follows from (2.5) that $\|u_n - \varphi(u_n)1\|^\sharp_\varphi \to 0$. This means that $\mathrm{Ad}\, u_n \to \mathrm{id}$ in $\mathrm{Aut}(P^I \rtimes \Lambda)$. Then also $\sigma^\varphi_{t_n} \to \mathrm{id}$ in $\mathrm{Aut}(P^I \rtimes \Lambda)$. Restricting $\sigma^\varphi_{t_n}$ to one copy of P, it follows that $t_n \to 0$ in τ.

Conversely, assume that $P^I \rtimes \Lambda$ is full. We prove that (i) holds. Let $y_n \in L\Lambda$ be a bounded central sequence satisfying $\|y_n - E_{L(\mathrm{Stab}\, i)}(y_n)\|_2 \to 0$ for every $i \in I$. Denote by $P_0 \subset P$ the set of elements $b \in P$ such that the right multiplication with b is a bounded operator on $L^2(P,\phi)$. Define M_0 as the linear span of all $P_0^{\mathcal{F}} u_g$, $\mathcal{F} \subset I$ finite and $g \in \Lambda$. It follows that $\|y_n a - a y_n\|^\sharp_\varphi \to 0$ for all $a \in M_0$. Since moreover $y_n \varphi = \varphi y_n$ for all n, it follows from [Con74, 2.8 and 3.1] that $\|y_n - \tau(y_n)1\|_2 \to 0$. □

Notes on Chapter 2

Section 2.1 is a general introduction of the type classification of von Neumann algebras, and all of the described concepts are well known in the area, see e.g. [Tak02]. Section 2.2 appeared earlier in [VV14] and [Ver15]. Section 2.3 contains both known results in Sections 2.3.1 to 2.3.2, and new results in Section 2.3.3. Section 2.3.3 is based on Section 4 of [VV14], but also includes [Ver15, Lemma 2.2] as Lemma 2.13. Lemmas 2.25, 2.28 and 2.29 in Section 2.4 appeared in [Ver15], while Lemma 2.26 appeared in [VV14]. Finally, Section 2.5 contains Lemmas 2.33 and 2.34 from [VV14], and Lemma 2.30 is a generalized version of [VV14, Lemma 2.5].

Chapter 3

Bernoulli crossed products built from almost periodic states

This and the next chapter are devoted to the classification of the noncommutative Bernoulli crossed products, which we introduced in Section 2.5. In this chapter, based on our joint work with Stefaan Vaes [VV14], we always assume that the base algebra (P, ϕ) is amenable and carries an *almost periodic state*, i.e. a state ϕ for which the modular operator Δ_ϕ on $L^2(P, \phi)$ is diagonalizable. Throughout the chapter, we denote by $\Gamma(P, \phi)$ the subgroup of $\mathbb{R}_0^+$ generated by the point spectrum of the modular operator Δ_ϕ. The main goal of this chapter, is to prove Theorem A:

Theorem A. *The set of factors*

$$\Big\{ (P,\phi)^\Lambda \rtimes \Lambda \;\Big|\; \begin{array}{l} P \textit{ a nontrivial amenable factor with normal faithful almost} \\ \textit{periodic state } \phi, \textit{ and } \Lambda = \Sigma \star \Upsilon \textit{ a free product of an infinite} \\ \textit{amenable group } \Sigma \textit{ and a nontrivial countable group } \Upsilon \end{array} \Big\}$$

is exactly classified, up to isomorphism, by $\Gamma(P, \phi) \subset \mathbb{R}_0^+$ *and the isomorphism class of* Λ.

To prove Theorem A, there is an 'isomorphism part' in which we prove that the factors are isomorphic when the invariants are the same, and a 'non-isomorphism part' in which we recover the invariants from the factor. Although the proof of the 'isomorphism part' is very specific to Bernoulli actions and lacks an immediate generalization, the 'non-isomorphism part' is a consequence of a wider phenomenom : for any outer state-preserving action $\Lambda \curvearrowright (P, \phi)$ on a von Neumann algebra equipped with a normal faithful almost periodic state, we have the following rigidity result, provided that the centralizer P_ϕ is a factor and the resulting crossed product is a full factor. The class $\mathcal{C}$ of countable groups was introduced in Definition 2.18 and by Remark 2.19, it contains all nonelementary hyperbolic groups, as well as all nontrivial free product groups.

Theorem (See Theorem 3.9)**.** *For $i = 0, 1$, let Λ_i be icc groups in the class $\mathcal{C}$. Let (P_i, ϕ_i) be amenable factors equipped with normal faithful almost periodic states having a factorial discrete decomposition, or equivalently, for which the centralizer $(P_i)_{\phi_i}$ is a factor. Let $\Lambda_i \curvearrowright (P_i, \phi_i)$ be outer, state-preserving actions such that the crossed products $P_i \rtimes \Lambda_i$ are full.*

Then the following two statements are equivalent.

(i) *The crossed products $P_i \rtimes \Lambda_i$ are isomorphic.*

(ii) *The groups Λ_0, Λ_1 are isomorphic, the point spectra of Δ_{ϕ_0} and Δ_{ϕ_1} coincide and there exists a projection $p \in (P_1)_{\phi_1}$, equal to 1 if the ϕ_i are traces, such that $\Lambda_0 \curvearrowright (P_0, \phi_0)$ is cocycle conjugate to the reduced cocycle action $(\Lambda_1 \curvearrowright P_1)^p$ through a state-preserving isomorphism, modulo the group isomorphism $\Lambda_0 \cong \Lambda_1$.*

The above theorem can also be applied to Bernoulli crossed products, as well as to two-sided Bernoulli crossed products. In particular, it provides a partial proof of Theorems D and E, as it classifies the subfamilies constructed with almost periodic states, see Section 3.4.

The proofs of the above classification theorems for type III factors with almost periodic states both make use of the modular theory of Connes-Takesaki and the so-called discrete decomposition of an almost periodic factor, see Section 3.1.1. In Lemma 3.5, we provide a technical lemma that relates cocycle conjugacy of state-preserving actions on almost periodic factors (P, ϕ) with cocycle conjugacy of the associated actions on the discrete decomposition of (P, ϕ).

In Section 3.2, we establish the isomorphism part in Theorem A first for Bernoulli actions of infinite amenable groups Σ using [Ocn85] and then co-induce to $\Sigma \star \Upsilon$ by a noncommutative version of [Bow09]. To prove Theorem 3.9 and the non-isomorphism part of Theorem A, we first pass to the discrete decomposition and then use the main results of [PV11, PV12, Ioa12] providing unique crossed product decomposition theorems for factors of the form $R \rtimes \Lambda$, where $\Lambda \curvearrowright R$ is an outer action on the hyperfinite II_1 factor R (see also Section 2.3).

3.1 Preliminaries on almost periodic states

3.1.1 Connes-Takesaki's discrete decomposition for almost periodic factors

A normal semifinite faithful (n.s.f.) weight φ on a von Neumann algebra M is called *almost periodic* if the modular operator Δ_φ on $L^2(M,\varphi)$ is diagonalizable. In this section, we develop modular theory for almost periodic weights as was introduced by Connes in [Con73, Con74], and show in particular that type III factors M having an almost periodic weight can be decomposed as $M = N \rtimes \Gamma$ with N a semifinite von Neumann algebra and Γ a discrete group with a trace scaling action on N.

Fix an almost periodic n.s.f. weight φ on a von Neumann algebra M. Let $\Gamma \subset \mathbb{R}_0^+$ be the subgroup generated by the point spectrum of Δ_φ, and endowed with the discrete topology. Let $\hat{\iota} : \Gamma \hookrightarrow \mathbb{R}_0^+$ denote the inclusion map, and let $G = \hat{\Gamma}$. Consider $\mathbb{R}_0^+$ to be the dual of $\mathbb{R}$ under the pairing $\langle t, \mu \rangle = \mu^{\mathrm{i}t}$ for $t \in \mathbb{R}$, $\mu \in \mathbb{R}_0^+$. Then there is a continuous group homomorphism $\iota : \mathbb{R} \to G$ determined by $\langle \iota(t), \gamma \rangle = \langle t, \hat{\iota}(\gamma) \rangle$, for $\gamma \in \Gamma, t \in \mathbb{R}$. Since $\hat{\iota}$ is injective, the image of ι is dense in G.

The next lemma shows that for almost periodic weights, the modular action can be extended to a compact group action, as was shown by Connes [Con73, Lemma 3.7.3]. We include a proof for the convenience of the reader.

Lemma 3.1 ([Con73, Lemma 3.7.3])**.** *There is a unique continuous action σ of G on M such that $\varphi \circ \sigma_s = \varphi$ for $s \in G$ and such that $\sigma_{\iota(t)} = \sigma_t^\varphi$ for all $t \in \mathbb{R}$.*

Conversely, the existence of a continuous homomorphism ι from $\mathbb{R}$ to a compact group G with dense image and a continuous group action $\sigma : G \curvearrowright M$, such that $\sigma_{\iota(t)} = \sigma_t^\varphi$ for all $t \in \mathbb{R}$, implies that φ is almost periodic.

Proof. Because φ is almost periodic, we find an orthogonal family of projections $(p_\gamma)_{\gamma\in\Gamma}$ of $L^2(M,\varphi)$ such that $\sum_\gamma p_\gamma = 1$ and $\Delta_\varphi = \sum_\gamma \hat{\iota}(\gamma) p_\gamma$. Write now for every $s \in G$, $V_s = \sum_\gamma \langle s, \gamma \rangle p_\gamma$, then we have for every $t \in \mathbb{R}$ that

$$V_{\iota(t)} = \sum_\gamma \langle \iota(t), \gamma \rangle p_\gamma = \sum_\gamma \langle t, \hat{\iota}(\gamma) \rangle p_\gamma = \sum_\gamma \hat{\iota}(\gamma)^{\mathrm{i}t} p_\gamma = \Delta^{\mathrm{i}t},$$

The map $s \mapsto V_s$ is strongly continuous, and hence $V_s \pi_\varphi(x) V_s^\star \in \pi_\varphi(M)$ for all $s \in G$. Define now $\sigma_s \in \mathrm{Aut}(M)$ by $\sigma_s(x) = V_s x V_s^\star$ for every x. By construction, $\sigma_{\iota(t)} = \sigma_t^\varphi$ for all $t \in \mathbb{R}$ and from the density of $\iota(\mathbb{R})$ in G, it follows that $\varphi \circ \sigma_s = \varphi$ for $s \in G$.

Assume now that φ is any n.s.f. weight on M, such that σ_t^φ extends to a compact group action $\sigma : G \curvearrowright M$ through a continuous homomorphism $\iota : \mathbb{R} \to G$ with dense image. Then also the action of $\mathbb{R}$ on $L^2(M, \varphi)$ by $\Delta_\varphi^{\mathbf{i}t}$ extends to a continuous unitary representation $g \mapsto U_g$ of G on $L^2(M, \varphi)$. Note that G is abelian, and denote by $\Gamma \subset \mathbb{R}_0^+$ the dual group of G with pairing $\langle \iota(t), \gamma \rangle = \gamma^{\mathbf{i}t}$ for $t \in \mathbb{R}, \gamma \in \Gamma$. For every $\gamma \in \Gamma$ and $\xi \in L^2(M, \varphi)$, put

$$P_\gamma(\xi) = \int_G \overline{\langle g, \gamma \rangle} U_g \xi dg,$$

and remark that $U_g(P_\gamma(\xi)) = \langle g, \gamma \rangle P_\gamma(\xi)$ for all $g \in G$, and thus $\Delta_\varphi(P_\gamma(\xi)) = \gamma P_\gamma(\xi)$. Fix $\xi, \eta \in L^2(M, \varphi)$ and note that $\gamma \mapsto \langle P_\gamma(\xi), \eta \rangle$ is the Fourier transform of the function $g \mapsto \langle U_g(\xi), \eta \rangle$. In particular, we obtain by the inverse transform that for all $g \in G$,

$$\langle U_g(\xi), \eta \rangle = \sum_{\gamma \in \Gamma} \langle g, \gamma \rangle \langle P_\gamma(\xi), \eta \rangle.$$

Taking $g = e$, we see that $\xi = \sum_{\gamma \in \Gamma} P_\gamma(\xi)$, which proves that Δ_φ is diagonalizable. $\square$

For every $\gamma \in \Gamma$, we denote the Arveson-Connes spectral subspace (see also section 2.1 of [Con73] and section XI.1 of [Tak03a], whose conventions we follow) of γ by

$$M_{\varphi,\gamma} = \{x \in M \mid \forall g \in G : \sigma_g(x) = \langle g, \gamma \rangle x\}.$$

The second part of Lemma 3.1 shows that for every $x \in M$ and every finite linear combination $\xi \in L^2(M, \varphi)$ of eigenvectors for Δ_φ, $x\xi = \sum_{\gamma \in \Gamma} E_\gamma(x)\xi$, with

$$E_\gamma(x) = \int_G \overline{\langle g, \gamma \rangle} \sigma_g(x) dg \quad \in M_{\varphi,\gamma}.$$

Actually, one can show that the linear span of $\bigcup_{\gamma \in \Gamma} M_{\varphi,\gamma}$ is a strongly dense $\star$-subalgebra of M [Dyk94, Lemma 1.2.3]. Moreover, it follows from Lemma 2.27 that $M_{\varphi,\gamma} \neq \{0\}$ if and only if $\gamma \in$ point spectrum Δ_φ.

If φ is a state, then

$$M_{\varphi,\gamma} = \{x \in M \mid \Delta_\varphi \eta_\varphi(x) = \hat{\iota}(\gamma) \eta_\varphi(x)\},$$

and for all $a \in M_{\varphi,\gamma}$ and $b \in M$ we have $\varphi(ab) = \hat{\iota}(\gamma)\varphi(ba)$, as

$$\langle b, a^\star \rangle_\varphi = \langle Sb^\star, Sa \rangle_\varphi = \langle S^\star Sa, b^\star \rangle_\varphi = \langle \Delta_\varphi a, b^\star \rangle_\varphi = \hat{\iota}(\gamma) \langle a, b^\star \rangle_\varphi.$$

The crossed product $M \rtimes_\sigma G$ of M with the action σ is called the *discrete core* of M. We denote by $\pi_\sigma : M \to M \rtimes_\sigma G$ the canonical embedding, and by $(\lambda(s))_{s\in G}$ the canonical group of unitaries such that

$$\pi_\sigma(\sigma_s(x)) = \lambda(s)\pi_\sigma(x)\lambda(s)^\star.$$

We denote by $\hat\sigma$ the *dual action* of Γ on $M \rtimes_\sigma G$, given by

$$\hat\sigma_\gamma(\pi_\sigma(x)) = \pi_\sigma(x) \text{ for all } x \in M \quad \text{ and } \quad \hat\sigma_\gamma(\lambda(s)) = \overline{\langle s, \gamma\rangle}\lambda(s) \text{ for all } s \in G.$$

The dynamical system $(M \rtimes_\sigma G, \Gamma, \hat\sigma)$ is the *discrete decomposition* associated to φ. We have by Takesaki duality (see Theorem 2.1) combined with Fourier transform that

$$(M \rtimes_\sigma G) \rtimes_{\hat\sigma} \Gamma \cong M \mathbin{\overline{\otimes}} B(\ell^2(\Gamma))$$

by the isomorphism $\Phi : (M \rtimes_\sigma G) \rtimes_{\hat\sigma} \Gamma \to M \mathbin{\overline{\otimes}} B(\ell^2(\Gamma))$ given by

$$\begin{aligned} \Phi\big(\pi_{\hat\sigma} \circ \pi_\sigma(x)\big) &= x \otimes \lambda_\gamma^\star, && \gamma \in \Gamma, \quad x \in M_{\varphi,\gamma}, \\ \Phi\big(\pi_{\hat\sigma} \circ \lambda(s)\big) &= 1 \otimes M_{\langle s,\cdot\rangle}^\star, && s \in G, \\ \Phi\big(\lambda(\gamma)\big) &= 1 \otimes \lambda_\gamma^\star. \end{aligned} \tag{3.1}$$

Here $M_{\langle s,\cdot\rangle}$ is the multiplication operator $\delta_\gamma \mapsto \langle s, \gamma\rangle\delta_\gamma$, while λ_γ is the translation operator $\delta_{\gamma'} \mapsto \delta_{\gamma\gamma'}$, and $(\lambda(\gamma))_{\gamma\in\Gamma}$ is the canonical group of unitaries in the second crossed product.

The bidual action $\hat{\hat\sigma}$ of G on $(M \rtimes_\sigma G) \rtimes_{\hat\sigma} \Gamma$ corresponds under Φ to the action $s \mapsto \sigma_s \otimes \operatorname{Ad} M_{\langle s,\cdot\rangle}$ on $M \mathbin{\overline{\otimes}} B(\ell^2(\Gamma))$. It follows that $M \rtimes_\sigma G$ is a factor if and only if M_φ is a factor, and in this case, the point spectrum of Δ_φ is already a group. A proof of this fact can be found in e.g. [Dyk94, Proposition 2.12], but for the convenience of the reader, we also include a proof.

Lemma 3.2 ([Dyk94, Proposition 2.12])**.** *$M \rtimes_\sigma G$ is a factor if and only if M_φ is a factor. In this case, the point spectrum of Δ_φ is already a group.*

Proof. Suppose that M_φ is a factor. Let Γ' be the subset of elements $\gamma \in \Gamma$ such that $M_{\varphi,\gamma} \neq \{0\}$. Since $M_{\varphi,\gamma} \neq \{0\}$ implies the existence of a partial isometry $v \in M_{\varphi,\gamma}$ with $vv^\star = 1$ or $v^\star v = 1$, we see that Γ' is a subgroup, hence $\Gamma' = \Gamma$. Denote by $p_\gamma \in B(\ell^2(\Gamma))$ the projection onto δ_γ, and by e_γ the canonical partial isometry with left support p_γ and right support p_e. If $v \in M_{\varphi,\gamma}$ is a partial isometry, we see that $v^\star \otimes e_\gamma \in M \mathbin{\overline{\otimes}} B(\ell^2(\Gamma))$ is invariant under the bidual action. Put $N = (M \mathbin{\overline{\otimes}} B(\ell^2(\Gamma)))^{\sigma_s \otimes \operatorname{Ad} M_{\langle s,\cdot\rangle}}$ and note that $(1 \otimes p_\gamma)N(1 \otimes p_\gamma) = M_\varphi \otimes p_\gamma$ is a factor, hence multiplying $v^\star \otimes e_\gamma$ with partial isometries of $(1 \otimes p_\gamma)N(1 \otimes p_\gamma)$ on the left and with partial isometries of $(1 \otimes p_e)N(1 \otimes p_e)$ on the right, we find partial isometries $w_i \in N$ such that $\bigvee_i w_i w_i^\star = 1 \otimes p_\gamma$ and $\bigvee_i w_i^\star w_i = 1 \otimes p_e$. It follows that for all $\gamma \in \Gamma$, the central support of $1 \otimes p_\gamma$ in N is 1. This implies that N is a factor: let $z \in \mathcal{Z}(N)$ be a nonzero projection, then necessarily $z = \sum_\gamma c_\gamma (1 \otimes p_\gamma)$ for $c_\gamma \in \{0, 1\}$ as $(1 \otimes p_\gamma)N(1 \otimes p_\gamma)$ is a factor. But for all $\gamma \in \Gamma$, we have that $z(1 \otimes p_\gamma) \neq 0$ as $1 \otimes p_\gamma$ has central support 1, hence $c_\gamma = 1$ and $z = 1$. Since $N \cong M \rtimes G$ by the Takesaki duality, it follows that $M \rtimes G$ is a factor.

Now assume that $M \rtimes G$ is a factor, then by the Takesaki duality, $N := (M \mathbin{\overline{\otimes}} B(\ell^2(\Gamma)))^{\sigma_s \otimes \operatorname{Ad} M_{\langle s, \cdot \rangle}}$ is a factor. Note that $1 \otimes p_e \in 1 \otimes \ell^\infty(\Gamma) \subset N$, and hence $(1 \otimes p_e)N(1 \otimes p_e) \cong M_\varphi$ is a factor. $\square$

If $M \rtimes_\sigma G$ is a factor, we say that M has a *factorial discrete decomposition.* Defining the weight $\tilde{\operatorname{Tr}}_\varphi$ on $(M \rtimes_\sigma G) \rtimes_{\hat{\sigma}} \Gamma$ by $\tilde{\operatorname{Tr}}_\varphi = \big(\varphi \otimes \operatorname{Tr}(M_{\hat{\imath}} \,\cdot)\big) \circ \Phi$, with $M_{\hat{\imath}}$ the multiplication operator associated to the unbounded positive function $\hat{\imath}$ on Γ, it is easy to see that $\tilde{\operatorname{Tr}}_\varphi$ is an almost periodic weight, and that the bidual action $\hat{\hat{\sigma}}$ is exactly the modular action for $\tilde{\operatorname{Tr}}_\varphi$. In particular, the restriction $\tilde{\operatorname{Tr}}_\varphi|_{M \rtimes G}$ to $M \rtimes_\sigma G$, the fixed point set under $\hat{\hat{\sigma}}$, yields a n.s.f. trace $\operatorname{Tr}_\varphi$ on $M \rtimes_\sigma G$ satisfying

$$\operatorname{Tr}_\varphi \circ \hat{\sigma}_\gamma = \hat{\imath}(\gamma)^{-1} \operatorname{Tr}_\varphi \text{ for all } \gamma \in \Gamma.$$

We need the following probably well known lemma.

Lemma 3.3. *Let (M, φ) be a factor equipped with an almost periodic normal faithful state having a factorial discrete decomposition $M \rtimes_\sigma G$. Let $\alpha \in \operatorname{Aut}(M, \varphi)$ be a state-preserving automorphism and denote by $\widetilde{\alpha} \in \operatorname{Aut}(M \rtimes_\sigma G)$ its canonical extension satisfying $\widetilde{\alpha}(\lambda(s)) = \lambda(s)$ for all $s \in G$.*

Then, the following statements are equivalent.

(i) *$\widetilde{\alpha}$ is an inner automorphism.*

(ii) *The restriction of α to M_φ is inner.*

(iii) *There exists a unitary $v \in M_\varphi$ and a group element $s \in G$ such that $\alpha = \operatorname{Ad} v \circ \sigma_s$.*

Proof. (i) $\Rightarrow$ (ii). Assume that $\widetilde{\alpha}$ is an inner automorphism. Denote by $p \in L(G)$ the minimal projection corresponding to the identity element in $\hat{G}$, so that $\lambda(s)p = p$ for all $s \in G$. Since $\widetilde{\alpha}(p) = p$, also the restriction of $\widetilde{\alpha}$ to $p(M \rtimes_\sigma G)p$ is inner. But $p(M \rtimes_\sigma G)p$ can be identified with M_φ and we conclude that (ii) holds.

(ii) $\Rightarrow$ (iii). Take $v \in \mathcal{U}(M_\varphi)$ such that $\alpha(x) = vxv^\star$ for all $x \in M_\varphi$. Define $\beta \in \mathrm{Aut}(M, \varphi)$ given by $\beta(x) = v^\star\alpha(x)v$ for all $x \in M$. We have to prove that $\beta = \sigma_s$ for some $s \in G$. Note that β is state-preserving and that $\beta(x) = x$ for all $x \in M_\varphi$. As above, consider $\Gamma = \hat{G}$ and fix $\gamma \in \Gamma$. Choose a nonzero partial isometry w in the spectral subspace $M_{\varphi,\gamma}$. Denote $p = w^\star w$ and note that $p \in M_\varphi$. Since β is state-preserving, we have that β commutes with the automorphisms σ_s, $s \in G$. Therefore, $w^\star\beta(w) \in pM_\varphi p$. Let $a \in pM_\varphi p$ be arbitrary. Then also $waw^\star \in M_\varphi$ and we get that

$$w^\star\beta(w)\, a = w^\star\beta(wa) = w^\star\beta(waw^\star\, w) = w^\star\, waw^\star\, \beta(w) = a\, w^\star\beta(w)\ .$$

Since $pM_\varphi p$ is a factor, we conclude that $w^\star\beta(w) = s(\gamma)p$ for some scalar $s(\gamma) \in \mathbb{T}$. Since M_φ is a factor, the linear span of $M_\varphi w M_\varphi$ equals $M_{\varphi,\gamma}$. It follows that $\beta(a) = s(\gamma)a$ for all $\gamma \in \Gamma$ and $a \in M_{\varphi,\gamma}$. We conclude that $s : \Gamma \to \mathbb{T}$ is a character. So, $s \in G$ and $\beta = \sigma_s$.

(iii) $\Rightarrow$ (i). If $\alpha = \mathrm{Ad}\, v \circ \sigma_s$ for some $v \in \mathcal{U}(M_\varphi)$ and $s \in G$, then $\widetilde{\alpha} = \mathrm{Ad}(v\lambda(s))$. □

3.1.2 State preserving actions and the discrete core

Every state-preserving action on a factor (P, ϕ) equipped with an almost periodic state induces an action on the discrete core of (P, ϕ). We prove two elementary lemmas, one giving a criterion that this induced action is outer and one relating cocycle conjugacy of the actions on the core with cocycle conjugacy of the original actions.

Let (P, ϕ) be a factor, with a normal faithful almost periodic state ϕ. Let Λ be a countable group, and assume that we are given a state-preserving action $\Lambda \curvearrowright^\alpha (P, \phi)$. Denote by $\Gamma \subset \mathbb{R}_0^+$ the group generated by the point spectrum of Δ_ϕ, and let $G = \hat{\Gamma}$. Consider the crossed product $N = P \rtimes_\sigma G$ of P by the modular action σ of G. Then Γ acts on N by the dual action $\hat{\sigma}$, and since the actions α and σ commute, Λ also acts on N by $\tilde{\alpha}$:

$$\tilde{\alpha}_s\big(\pi_\sigma(x)\big) = \pi_\sigma\big(\alpha_s(x)\big), \qquad s \in \Lambda, \quad x \in P,$$

$$\tilde{\alpha}_s(\lambda(g)) = \lambda(g), \qquad g \in G.$$

A direct calculation shows that also $\tilde{\alpha}$ and $\hat{\sigma}$ commute. We thus find the action β of the countable group $\Lambda \times \Gamma$ on N, given by $\beta_{(s,\gamma)} = \tilde{\alpha}_s \circ \hat{\sigma}_\gamma$.

Lemma 3.4. *Under the above assumptions, suppose that (P, ϕ) has a factorial discrete decomposition N. If the group Λ has trivial center and the action $\Lambda \curvearrowright^\alpha P$ is outer, then also the action $\widetilde{\alpha}$ of Λ on $P \rtimes_\sigma G$ is outer.*

Proof. Denote by Λ_0 the group of all $s \in \Lambda$ for which $\widetilde{\alpha}_s$ is inner. Viewing both Λ and G as subgroups of the outer automorphism group $\mathrm{Out}(P)$, it follows from Lemma 3.3 that $\Lambda_0 = \Lambda \cap G$. Since Λ and G commute inside $\mathrm{Out}(P)$, it follows that Λ_0 is a subgroup of the center of Λ. Therefore, $\Lambda_0 = \{e\}$. □

Lemma 3.5. *Let (P_0, ϕ_0), (P_1, ϕ_1) be factors with normal faithful almost periodic states and factorial discrete decompositions. Assume that Δ_{ϕ_0} and Δ_{ϕ_1} have the same point spectrum $\Gamma = \hat{G}$. Let Λ be a countable group, with state-preserving actions $\Lambda \curvearrowright^{\alpha^i} (P_i, \phi_i)$.*

Assume that $\psi : P_0 \rtimes_{\sigma^0} G \to P_1 \rtimes_{\sigma^1} G$ is an isomorphism such that the induced actions $\Lambda \times \Gamma \curvearrowright^{\beta^i} P_i \rtimes_{\sigma^i} G$ are cocycle conjugate through ψ. Then there exists a projection $p \in (P_1)_{\phi_1}$ such that the cocycle actions α^0 and $(\alpha^1)^p$ are cocycle conjugate through a state-preserving isomorphism. If $\operatorname{mod}\psi \in \Gamma$, then we can take $p = 1$.

Proof. If $\Gamma = \{1\}$, there is nothing to prove, so we assume that $\Gamma \neq \{1\}$. Write $N_i = P_i \rtimes_{\sigma^i} G$ and let $\psi : N_0 \to N_1$ be the isomorphism given in the theorem. Write $\kappa = \operatorname{mod}\psi$. Since we may replace ψ by $\psi \circ \hat{\sigma}^0_\gamma$ for any $\gamma \in \Gamma$, we may assume that $\kappa \leq 1$ and that $\kappa = 1$ if the original $\operatorname{mod}\psi$ belonged to Γ.

Since ψ is a cocycle conjugacy between the actions $\Lambda \times \Gamma \curvearrowright N_i$, we can first extend ψ to an isomorphism $\widetilde{\psi} : N_0 \rtimes_{\hat{\sigma}^0} \Gamma \to N_1 \rtimes_{\hat{\sigma}^1} \Gamma$ and then observe that $\widetilde{\psi}$ remains a cocycle conjugacy for the natural actions of Λ. We identify $N_i \rtimes_{\hat{\sigma}^i} \Gamma$ with $P_i \mathbin{\overline{\otimes}} B(\ell^2(\Gamma))$ through the Takesaki duality isomorphism given in (3.1). Under this identification, the dual weight becomes $\phi_i \otimes \omega$ where ω is the weight on $B(\ell^2(\Gamma))$ given by $\omega = \mathrm{Tr}(M_{\hat{\iota}} \,\cdot)$. The Λ-action is now given by $\alpha^i \otimes \mathrm{id}$ and N_i corresponds to the subalgebra $(P_i \mathbin{\overline{\otimes}} B(\ell^2(\Gamma)))_{\phi_i \otimes \omega}$. From now on, we denote the latter as N_i.

Therefore under these identifications, $\widetilde{\psi}$ becomes an isomorphism

$$\Psi : P_0 \mathbin{\overline{\otimes}} B(\ell^2(\Gamma)) \to P_1 \mathbin{\overline{\otimes}} B(\ell^2(\Gamma))$$

with the following properties: Ψ is a cocycle conjugacy for the actions $\alpha^i \otimes \mathrm{id}$, we have that $(\phi_1 \otimes \omega) \circ \Psi = \kappa\,(\phi_0 \otimes \omega)$, and $\Psi(N_0) = N_1$.

Let $e \in B(\ell^2(\Gamma))$ be the projection onto $\mathbb{C}\delta_1 \subset \ell^2(\Gamma)$. Then $1 \otimes e$ is a projection in N_0 with weight 1. Therefore, $\Psi(1 \otimes e)$ is a projection in N_1 with weight $\kappa \leq 1$. Since N_1 is a II_∞ factor, we can replace Ψ by $(\operatorname{Ad} v) \circ \Psi$ for some unitary $v \in \mathcal{U}(N_1)$ and assume that $\Psi(1 \otimes e) = p \otimes e$ where p is a projection in $(P_1)_{\phi_1}$ with $\phi_1(p) = \kappa$. Restricting Ψ to

$$P_0 = (1 \otimes e)(P_0 \mathbin{\overline{\otimes}} B(\ell^2(\Gamma)))(1 \otimes e)$$

gives the conclusion of the lemma. □

3.2 Isomorphism results for type III Bernoulli crossed products

In [Ocn85], Ocneanu obtained a complete classification up to cocycle conjugacy of actions of amenable groups on the hyperfinite II_1 or II_∞ factor. We deduce as a corollary of Ocneanu's work a classification result for actions of amenable groups on amenable factors equipped with an almost periodic state. We work with state-preserving actions and look for state-preserving cocycle conjugacies.

It is essential for us to work in a state-preserving setting throughout. The reason is that in a second step, we develop a noncommutative version of Bowen's co-induction method [Bow09] to pass from cocycle conjugacy of noncommutative Bernoulli Σ-actions to cocycle conjugacy of Bernoulli $\Sigma * \Upsilon$-actions. This method only works if the original cocycle conjugacy is state-preserving. At the end of this section, we then find the 'isomorphism part' of Theorem A.

Theorem 3.6. *Let (M_0, ϕ_0), (M_1, ϕ_1) be nontrivial amenable factors, with normal faithful almost periodic states having factorial discrete decompositions. Let Σ be a countably infinite amenable group, with state-preserving actions $\Sigma \curvearrowright^{\alpha^i} (M_i, \phi_i)$, for $i = 0, 1$. Assume that the restrictions of α^i to $(M_i)_{\phi_i}$ are outer.*

The actions α^0 and α^1 are cocycle conjugate through a state-preserving isomorphism if and only if the point spectra of Δ_{ϕ_i} coincide.

Proof. One implication being obvious, assume that the point spectra of Δ_{ϕ_i} coincide. Denote by G the canonically associated compact group with the modular automorphism groups $(\sigma^i_s)_{s \in G}$. When $G = \{1\}$, the ϕ_i are traces and the theorem is exactly [Ocn85, Corollary 1.4]. So we may assume that the ϕ_i are not traces. Since the actions α^i are state-preserving, they canonically extend to actions $\widetilde{\alpha}^i$ of Σ on $N_i = M_i \rtimes_{\sigma^i} G$. Note that the action $\widetilde{\alpha}^i$ commutes with the dual action $(\hat{\sigma}_\gamma)_{\gamma \in \Gamma}$ of $\Gamma = \hat{G}$. Both combine into an action $(\beta^i_{(\sigma,\gamma)})_{(\sigma,\gamma) \in \Sigma \times \Gamma}$ of the amenable group $\Sigma \times \Gamma$ on the hyperfinite II_∞ factor N_i.

We want to apply [Ocn85, Theorem 2.9] to get that β^0 and β^1 are cocycle conjugate. For this, it suffices to check that the β^i are outer actions that scale the trace in the same way. The latter follows because $\operatorname{mod} \beta^i_{(\sigma,\gamma)} = \hat{\imath}(\gamma)^{-1}$. To prove the former, assume that $\beta^i_{(\sigma,\gamma)}$ is inner. Since this automorphism scales the trace with the factor $\hat{\imath}(\gamma)^{-1}$, we get that $\gamma = 1$. So, $\widetilde{\alpha}^i_\sigma$ is inner. It then follows from Lemma 3.3 that the restriction of α^i_σ to $(M_i)_{\phi_i}$ is inner. Therefore, $\sigma = e$.

We claim that the actions β^i are cocycle conjugate through a trace-preserving isomorphism. To prove this claim, denote by R_∞ the hyperfinite II_∞ factor. Also the action $\beta^1 \otimes \mathrm{id}$ on $N_1 \mathbin{\overline{\otimes}} R_\infty$ is outer and scales the trace in the same way as the actions β^i. Applying twice [Ocn85, Theorem 2.9], we find a cocycle conjugacy $\psi_0 : N_0 \to N_1 \mathbin{\overline{\otimes}} R_\infty$ between β^0 and $\beta^1 \otimes \mathrm{id}$, as well as a cocycle conjugacy $\psi_1 : N_1 \mathbin{\overline{\otimes}} R_\infty \to N_1$ between $\beta^1 \otimes \mathrm{id}$ and β^1. Choosing an automorphism $\theta \in \mathrm{Aut}(R_\infty)$ with $\mathrm{mod}\,\theta = \mathrm{mod}\,\psi_0 \cdot \mathrm{mod}\,\psi_1$, it follows that $\psi = \psi_1 \circ (\mathrm{id} \otimes \theta^{-1}) \circ \psi_0$ is a trace-preserving cocycle conjugacy between β^0 and β^1. The conclusion follows from Lemma 3.5. $\square$

We now recall the construction of co-induced actions, which we use to lift the result of Theorem 3.6 to Bernoulli actions of free product groups. Let $\Sigma \curvearrowright^\alpha (M, \phi)$ be a countable group acting on a von Neumann algebra by state-preserving automorphisms. Assume that Σ is a subgroup of a countable discrete group Λ. Choose a map $r : \Lambda \to \Sigma$ such that $r(g\sigma) = r(g)\sigma$ for all $g \in \Lambda, \sigma \in \Sigma$ and such that $r(e) = e$. We have the associated 1-cocycle $\Omega : \Lambda \times \Lambda/\Sigma \to \Sigma$ for the left action of Λ on Λ/Σ, given by $\Omega(g, h\Sigma) = r(gh)r(h)^{-1}$ for all $g, h \in \Lambda$. Let $M^{\Lambda/\Sigma}$ be the tensor product of M indexed by Λ/Σ, with respect to the state ϕ, and denote by $\pi_{h\Sigma} : M \to M^{\Lambda/\Sigma}$ the embedding of M at position $h\Sigma$, for $h \in \Lambda$. The formula

$$\Lambda \curvearrowright^\beta M^{\Lambda/\Sigma} \text{ where } \beta_g\big(\pi_{h\Sigma}(a)\big) = \pi_{gh\Sigma}\big(\alpha_{\Omega(g,h\Sigma)}(a)\big), \quad a \in M,\ g, h \in \Lambda,$$

yields a well defined state-preserving action of Λ on the tensor product $(M^{\Lambda/\Sigma}, \phi^{\Lambda/\Sigma})$, called the *co-induced action* of $\Sigma \curvearrowright (M, \phi)$ to Λ.

The following proposition provides a variant of Bowen's co-induction argument in [Bow09], with a proof in the spirit of [MRV11].

Proposition 3.7. *Let Σ and Υ be countable groups, and $(M_0, \phi_0), (M_1, \phi_1)$ be von Neumann algebras equipped with normal faithful states. Assume that $\Sigma \curvearrowright^{\alpha^i} (M_i, \phi_i)$ are state-preserving actions, for $i = 0, 1$. Denote $\Lambda = \Sigma * \Upsilon$, and let $\Lambda \curvearrowright^{\beta^i} M_i^{\Lambda/\Sigma}$ be the co-induced actions.*

If the actions α^i are cocycle conjugate through a state-preserving isomorphism, then also the co-induced actions β^i are cocycle conjugate through a state-preserving isomorphism.

Proof. Consider two cocycle conjugate actions α^i of Σ and their co-inductions β^i to $\Lambda = \Sigma * \Upsilon$, as in the statement of the theorem. Let $\psi : M_0 \to M_1$ be a state-preserving isomorphism and $(w_\sigma)_{\sigma\in\Sigma} \in (M_1)_{\phi_1}$ be a 1-cocycle for α^1 such that

$$\psi \circ \alpha^0_\sigma = \mathrm{Ad}\, w_\sigma \circ \alpha^1_\sigma \circ \psi, \qquad \forall \sigma \in \Sigma.$$

Consider the action $\tilde{\alpha}^0$ of Σ on M_1 given by $\tilde{\alpha}^0_\sigma = \psi \circ \alpha^0_\sigma \circ \psi^{-1}$, for $\sigma \in \Sigma$. The infinite tensor product $\psi^{\Lambda/\Sigma}$ is a state-preserving conjugacy between the co-induction of α^0 and the co-induction of $\tilde{\alpha}^0$. So, it is enough to show that the co-induced actions of $\tilde{\alpha}^0$ and α^1 to Λ are cocycle conjugate through a state-preserving isomorphism. This allows us to simplify the notations: we put $M = M_1$, $\alpha = \alpha^1$ and $\tilde{\alpha} = \tilde{\alpha}^0$. We then have

$$\tilde{\alpha}_\sigma = \operatorname{Ad} w_\sigma \circ \alpha_\sigma, \qquad \sigma \in \Sigma.$$

Take $\mathcal{I} \subset \Lambda = \Sigma * \Upsilon$ to be the left transversal of $\Sigma < \Lambda$ consisting of the trivial word e and the words ending in a letter of $\Upsilon - \{e\}$. Consider the map $r : \Lambda \to \Sigma$ defined by $r(g\sigma) = \sigma$ whenever $g \in \mathcal{I}$ and $\sigma \in \Sigma$, and let $\Omega : \Lambda \times \Lambda/\Sigma \to \Sigma$, $\Omega(g, h\Sigma) = r(gh)r(h)^{-1}$ be the associated 1-cocycle. Denote by β and $\tilde{\beta}$ the co-induced actions to Λ of α and $\tilde{\alpha}$ respectively, with respect to Ω. Remark that for $\sigma \in \Sigma$, $\upsilon \in \Upsilon$, $h \in \mathcal{I} - \{e\}$,

$$\Omega(\sigma, h\Sigma) = e \qquad \text{because } \sigma(\mathcal{I} - \{e\}) \subset \mathcal{I} - \{e\},$$

$$\Omega(\sigma, e\Sigma) = \sigma,$$

$$\Omega(\upsilon, h\Sigma) = e \qquad \text{because } \upsilon\mathcal{I} \subset \mathcal{I},$$

$$\Omega(\upsilon, e\Sigma) = e.$$

Since $\Lambda = \Sigma * \Upsilon$ is the free product of Υ and Σ, there is a unique 1-cocycle $W_g \in (M^{\Lambda/\Sigma})_{\phi^{\Lambda/\Sigma}}$, $g \in \Lambda$ for β given by

$$W_\sigma = \pi_{e\Sigma}(w_\sigma), \quad \sigma \in \Sigma, \qquad\qquad W_\upsilon = 1, \quad \upsilon \in \Upsilon.$$

We claim that $\tilde{\beta}_g = \operatorname{Ad} W_g \circ \beta_g$ for all $g \in \Lambda$, from which the lemma follows. It suffices to prove that this equation holds for all elements of Σ and Υ. Let $\sigma \in \Sigma$, then, for $h \in \mathcal{I} - \{e\}$, $a \in M$,

$$\tilde{\beta}_\sigma\big(\pi_{h\Sigma}(a)\big) = \pi_{\sigma h\Sigma}\big(\tilde{\alpha}_e(a)\big) = \pi_{\sigma h\Sigma}(a),$$

$$\operatorname{Ad} W_\sigma \circ \beta_\sigma\big(\pi_{h\Sigma}(a)\big) = \pi_{e\Sigma}(w_\sigma)\pi_{\sigma h\Sigma}\big(\alpha_e(a)\big)\pi_{e\Sigma}(w_\sigma^\star) = \pi_{\sigma h\Sigma}(a),$$

and also

$$\begin{aligned}\tilde{\beta}_\sigma\big(\pi_{e\Sigma}(a)\big) = \pi_{e\Sigma}\big(\tilde{\alpha}_\sigma(a)\big) &= \pi_{e\Sigma}\big(w_\sigma \alpha_\sigma(a) w_\sigma^\star\big) \\ &= W_\sigma \pi_{e\Sigma}\big(\alpha_\sigma(a)\big) W_\sigma^\star = \operatorname{Ad} W_\sigma \circ \beta_\sigma\big(\pi_{e\Sigma}(a)\big),\end{aligned}$$

which means that $\tilde{\beta}_\sigma(x) = \operatorname{Ad} W_\sigma \circ \beta_\sigma(x)$ for all $x \in M^{\Lambda/\Sigma}$. Let now $\upsilon \in \Upsilon$, and $h \in \mathcal{I}$, $a \in M$, then

$$\tilde{\beta}_\upsilon\big(\pi_{h\Sigma}(a)\big) = \pi_{\upsilon h\Sigma}\big(\tilde{\alpha}_e(a)\big) = \pi_{\upsilon h\Sigma}\big(a\big) = \pi_{\upsilon h\Sigma}\big(\alpha_e(a)\big) = \operatorname{Ad} W_\upsilon \circ \beta_\upsilon\big(\pi_{h\Sigma}(a)\big),$$

so $\tilde{\beta}_\upsilon = \operatorname{Ad} W_\upsilon \circ \beta_\upsilon$ as well. □

The previous two results now yield the following theorem, providing the 'isomorphism part' of Theorem A.

Theorem 3.8. *Suppose that (P_0, ϕ_0) and (P_1, ϕ_1) are two nontrivial amenable factors, equipped with normal faithful almost periodic states ϕ_0, ϕ_1. Let Σ be a countably infinite amenable group, and Υ any countable group. The following two statements are equivalent.*

(i) *The point spectra of Δ_{ϕ_0} and Δ_{ϕ_1} generate the same subgroup of $\mathbb{R}_0^+$.*

(ii) *With $\Lambda = \Sigma * \Upsilon$, the Bernoulli actions $\Lambda \curvearrowright (P_0, \phi_0)^\Lambda$ and $\Lambda \curvearrowright (P_1, \phi_1)^\Lambda$ are cocycle conjugate through a state-preserving isomorphism.*

Proof. If (ii) holds, it follows that $\Delta_{\phi_0^\Lambda}$ and $\Delta_{\phi_1^\Lambda}$ have the same point spectrum, which exactly means that (i) holds. We prove the implication (i) $\Rightarrow$ (ii).

Denote by ρ^i the Bernoulli action of Σ on P_i^Σ, for $i = 0, 1$. The factors P_0^Σ, P_1^Σ are amenable, and they have normal faithful almost periodic states and factorial discrete decompositions, by Lemma 2.26. Moreover, the point spectra of $\Delta_{\phi_0^\Sigma}$, $\Delta_{\phi_1^\Sigma}$ form the same subgroup of $\mathbb{R}_0^+$. Combining Lemmas 3.3 and 2.30, the Bernoulli actions of Σ on P_i^Σ are state-preserving and outer when restricted to the centralizer of ϕ_i^Σ. By Theorem 3.6, the Bernoulli actions of Σ on P_0^Σ and P_1^Σ are cocycle conjugate through a state-preserving isomorphism.

Note that whenever $\Sigma < \Lambda$ is a subgroup, the co-induced action of the Bernoulli action $\Sigma \curvearrowright (P_i, \phi_i)^\Sigma$ to Λ is exactly the Bernoulli action $\Lambda \curvearrowright (P_i, \phi_i)^\Lambda$. By Proposition 3.7, (ii) follows. $\square$

3.3 A non-isomorphism result for almost periodic crossed products

We prove the following general rigidity theorem for crossed products of groups in the class $\mathcal{C}$ acting by state-preserving automorphisms of an amenable factor with almost periodic state. This will be a crucial ingredient to distinguish between type III Bernoulli crossed products.

Theorem 3.9. *For $i = 0, 1$, let Λ_i be icc groups in the class $\mathcal{C}$. Let (P_i, ϕ_i) be amenable factors equipped with normal faithful almost periodic states having a factorial discrete decomposition. Let $\Lambda_i \curvearrowright (P_i, \phi_i)$ be outer, state-preserving actions such that the crossed products $P_i \rtimes \Lambda_i$ are full.*

Then the following two statements are equivalent.

(i) *The crossed products* $P_i \rtimes \Lambda_i$ *are isomorphic.*

(ii) *The groups* Λ_0, Λ_1 *are isomorphic, the point spectra of* Δ_{ϕ_0} *and* Δ_{ϕ_1} *coincide and there exists a projection* $p \in (P_1)_{\phi_1}$*, equal to* 1 *if the* ϕ_i *are traces, such that* $\Lambda_0 \curvearrowright (P_0, \phi_0)$ *is cocycle conjugate to the reduced cocycle action* $(\Lambda_1 \curvearrowright P_1)^p$ *through a state-preserving isomorphism, modulo the group isomorphism* $\Lambda_0 \cong \Lambda_1$.

Proof. Denote by G_i the natural compact groups acting by the modular automorphisms $(\sigma_s^i)_{s\in G_i}$ on P_i and on $P_i \rtimes \Lambda_i$. Write $N_i = P_i \rtimes G_i$. The discrete decomposition of $P_i \rtimes \Lambda_i$ is then given as $N_i \rtimes \Lambda_i$. Since the groups Λ_i have trivial center, it follows from Lemma 3.4 that the actions $\Lambda_i \curvearrowright N_i$ are outer. In particular, $P_i \rtimes \Lambda_i$ is a full factor with a factorial discrete decomposition. Therefore, $\mathrm{Sd}(P_i \rtimes \Lambda_i)$ equals the point spectrum of Δ_{ϕ_i}. In particular, the factor $P_i \rtimes \Lambda_i$ is of type II_1 if ϕ_i is a trace and of type III if ϕ_i is not a trace.

Assume first that (ii) holds. In the tracial case, the II_1 factors $P_i \rtimes \Lambda_i$ follow isomorphic. In the non-tracial case, it follows that $P_0 \rtimes \Lambda_0$ is isomorphic to $p(P_1 \rtimes \Lambda_1)p$. But in that case, the factors $P_i \rtimes \Lambda_i$ are of type III so that also $P_0 \rtimes \Lambda_0 \cong P_1 \rtimes \Lambda_1$. This means that (i) holds.

Assume next that (i) holds and let $\psi : P_0 \rtimes \Lambda_0 \to P_1 \rtimes \Lambda_1$ be a $\star$-isomorphism. In particular, the factors $P_i \rtimes \Lambda_i$ have the same Sd-invariant, so that the point spectra of Δ_{ϕ_i} coincide. In the case where the ϕ_i are traces, because the groups Λ_i are icc and belong to the class $\mathcal{C}$, it follows from Lemma 2.12 that $\psi(P_0)$ is unitarily conjugate to P_1. Since the actions $\Lambda_i \curvearrowright P_i$ are outer, this means that $\Lambda_0 \cong \Lambda_1$ and that the actions $\Lambda_i \curvearrowright P_i$ are cocycle conjugate.

So assume that the ϕ_i are not traces. Since the point spectra of Δ_{ϕ_i} coincide, it follows that $G_0 = G_1$ and we denote this compact group as G. By [Con74, Lemma 4.2], ψ is a cocycle conjugacy between the modular automorphism groups $(\sigma_s^0)_{s\in G}$ and $(\sigma_s^1)_{s\in G}$ and therefore extends to a $\star$-isomorphism $\Psi : M_0 \to M_1$ between the crossed products $M_i = (P_i \rtimes \Lambda_i) \rtimes G = N_i \rtimes \Lambda_i$. The assumption that (P_i, ϕ_i) is amenable with factorial discrete decomposition means that N_i is the hyperfinite II_∞ factor. We have seen in the first paragraph of the proof that the actions $\Lambda_i \curvearrowright N_i$ are outer. Since the actions $\Lambda_i \curvearrowright P_i$ are state-preserving, the action of Λ_i on N_i equals the identity on $L(G) \subset N_i$.

We claim that $\Psi(N_0)$ and N_1 are unitarily conjugate inside M_1. Take a projection $p_0 \in L(G)$ of finite trace. Then $\Psi(p_0)$ is a projection of finite trace in the II_∞ factor $N_1 \rtimes \Lambda_1$. After a unitary conjugacy of Ψ, we find a projection $p_1 \in L(G)$ of finite trace such that $\Psi(p_0) \leq p_1$. Since the projections p_i are Λ_i-invariant, we have

$$p_i M_i p_i = p_i N_i p_i \rtimes \Lambda_i \ .$$

The restriction of Ψ to $p_0 M_0 p_0$ thus yields a $\star$-isomorphism of $p_0 N_0 p_0 \rtimes \Lambda_0$ onto a corner of $p_1 N_1 p_1 \rtimes \Lambda_1$. Because the groups Λ_i are icc and belong to the class $\mathcal{C}$, it follows from Lemma 2.12 that $\Psi(p_0 N_0 p_0)$ is unitarily conjugate to a corner of N_1. Since the N_i are II_∞ factors, the claim follows.

By the claim in the previous paragraph, we can choose a unitary $u \in \mathcal{U}(M_1)$ such that $u\Psi(N_0)u^\star = \Psi(N_1)$. In particular, $\Lambda_0 \cong \Lambda_1$ and $\operatorname{Ad} u \circ \Psi$ is a cocycle conjugacy between $\Lambda_0 \curvearrowright N_0$ and $\Lambda_1 \curvearrowright N_1$.

Denote $\Gamma = \hat{G}$ and let $\hat{\sigma}^i$ be the dual, trace scaling action of Γ on $M_i = (P_i \rtimes \Lambda_i) \rtimes G$. By construction, $\Psi \circ \hat{\sigma}^0_\gamma = \hat{\sigma}^1_\gamma \circ \Psi$ for all $\gamma \in \Gamma$. Therefore, Ψ further extends to a $\star$-isomorphism $\widetilde{\Psi} : M_0 \rtimes \Gamma \to M_1 \rtimes \Gamma$ satisfying $\widetilde{\Psi}(u_\gamma) = u_\gamma$ for all $\gamma \in \Gamma$. Write $\Theta = \operatorname{Ad} u \circ \widetilde{\Psi}$. Note that we can view $M_i \rtimes \Gamma$ as $N_i \rtimes (\Lambda_i \times \Gamma)$, and that the actions $\Lambda_i \times \Gamma \curvearrowright N_i$ are outer because the action of Γ is trace scaling and the action of Λ_i is trace-preserving and outer. By construction,

$$\Theta(N_0) = N_1 \quad , \quad \Theta(N_0 \rtimes \Lambda_0) = N_1 \rtimes \Lambda_1 \quad \text{and} \quad \Theta(u_\gamma) \in (N_1 \rtimes \Lambda_1) u_\gamma \; .$$

This means that the restriction of Θ to N_0 is a cocycle conjugacy between the actions $\Lambda_i \times \Gamma \curvearrowright N_i$, modulo a group isomorphism $\delta : \Lambda_0 \times \Gamma \to \Lambda_1 \times \Gamma$ satisfying $\delta(\Lambda_0) = \Lambda_1$ and $\delta(e,\gamma) \in \Lambda_1 \times \{\gamma\}$. Since Λ_i has trivial center, this means that $\delta(g,\gamma) = (\delta_0(g),\gamma)$ for all $g \in \Lambda_0$, $\gamma \in \Gamma$ and a group isomorphism $\delta_0 : \Lambda_0 \to \Lambda_1$.

It now follows from Lemma 3.5 that there exists a projection $p \in (P_1)_{\phi_1}$ such that $\Lambda_0 \curvearrowright (P_0, \phi_0)$ is cocycle conjugate to the reduced cocycle action $(\Lambda_1 \curvearrowright P_1)^p$ through a state-preserving isomorphism, modulo the group isomorphism $\delta_0 : \Lambda_0 \to \Lambda_1$. So we have proved that (ii) holds. □

3.4 Proof of Theorem A and partial proofs of Theorems D and E

Theorem A follows from Theorem 3.8 and the following result. Also note that (iii) in the theorem below provides a partial proof of Theorem D, as it shows that the set of factors

$$\Big\{ (P,\phi)^\Lambda \rtimes \Lambda \;\Big|\; \begin{array}{l} \textit{P a nontrivial amenable factor with normal faithful almost} \\ \textit{periodic state } \phi\textit{, and } \Lambda \textit{ a direct product of two icc groups} \\ \textit{in the class } \mathcal{C} \end{array} \Big\}$$

is exactly classified, up to isomorphism, by the group Λ and the action $\Lambda \curvearrowright (P,\phi)^\Lambda$ up to a state-preserving conjugacy of the actions. Recall that $\Gamma(P,\phi)$

denotes the subgroup of $\mathbb{R}_0^+$ generated by the point spectrum of the modular operator Δ_ϕ.

Theorem 3.10. *For $i = 0, 1$, let (P_i, ϕ_i) be nontrivial amenable factors equipped with a normal faithful state. Let Λ_i be icc groups in the class $\mathcal{C}$. Assume that ϕ_0 is almost periodic and that $P_0^{\Lambda_0} \rtimes \Lambda_0$ is isomorphic with $P_1^{\Lambda_1} \rtimes \Lambda_1$. Then the following holds.*

(i) *Also ϕ_1 is almost periodic and $\Gamma(P_0, \phi_0) = \Gamma(P_1, \phi_1)$.*

(ii) *The groups Λ_i are isomorphic.*

(iii) *If Λ_0 is a direct product of two icc groups in the class $\mathcal{C}$, then the actions $\Lambda_i \curvearrowright P_i^{\Lambda_i}$ are conjugate through a state-preserving isomorphism.*

Proof. Since Λ_i belongs to the class $\mathcal{C}$, certainly Λ_i is nonamenable. By Lemma 2.34, the factors $P_i^{\Lambda_i} \rtimes \Lambda_i$ are full and their τ-invariant is the weakest topology that makes the map $t \mapsto \sigma_t^{\phi_i}$ continuous. Since the factors $P_i^{\Lambda_i} \rtimes \Lambda_i$ are isomorphic, they have the same τ-invariant and Sd-invariant. As ϕ_0 is almost periodic, there exists a compact group G and a continuous homomorphism $\iota : \mathbb{R} \to G$ with dense image such that σ^{ϕ_0} extends to a continuous action of G. Putting τ_G the weakest topology on $\mathbb{R}$ such that ι is continuous, this means that the identity map $(\mathbb{R}, \tau_G) \to (\mathbb{R}, \tau)$ is continuous. In particular, the action $\sigma^{\phi_1} : (\mathbb{R}, \tau_G) \curvearrowright P_1$ can be extended to a continuous group action of G, and ϕ_1 is almost periodic by Lemma 3.1. Write $\varphi_i = \phi_i^{\Lambda_i}$. Since the action $\Lambda_i \curvearrowright (P_i^{\Lambda_i})_{\varphi_i}$ is properly outer by Lemmas 3.3 and 2.30, the centralizers $(P_i^{\Lambda_i} \rtimes \Lambda_i)_{\varphi_i} = (P_i^{\Lambda_i})_{\varphi_i} \rtimes \Lambda_i$ are factors by Lemmas 2.9 and 2.26, and it follows that the Sd-invariant of $P_i^{\Lambda_i} \rtimes \Lambda_i$ equals the point spectrum of Δ_{φ_i}. We conclude that (i) holds.

By Lemma 2.26, the factors $(P_i^{\Lambda_i}, \varphi_i)$ have a factorial discrete decomposition. By Lemma 2.30, the actions $\Lambda_i \curvearrowright^{\alpha_i} P_i^{\Lambda_i}$ are outer. Above, we already mentioned that the crossed products are full factors. By Theorem 3.9, (ii) holds and we find a projection $p \in (P_0^{\Lambda_0})_{\varphi_0}$ such that the action $\Lambda_1 \curvearrowright P_1^{\Lambda_1}$ is cocycle conjugate to the reduction of $\Lambda_0 \curvearrowright P_0^{\Lambda_0}$ by p, modulo the isomorphism $\Lambda_0 \cong \Lambda_1$, through a generalized 1-cocycle $(w_g)_{g \in \Lambda_0}$ for the action $\Lambda_0 \curvearrowright P_0^{\Lambda_0}$ having support projection p.

Assume now that Λ_0 is moreover a direct product of two nonamenable groups. We apply Corollary A.3 to the generalized 1-cocycle $(w_g)_{g \in \Lambda_0}$. Note that the action $\Lambda_1 \curvearrowright P_1^{\Lambda_1}$ satisfies property (ii) in Lemma 2.10, hence it has no nontrivial finite-dimensional globally invariant subspaces. It follows from Corollary A.3 that $w_g = \chi(g) w \alpha_g^0(w^\star)$ for all $g \in \Lambda_0$, where $\chi : \Lambda_0 \to \mathbb{T}$ is a character and $w \in (P_0^{\Lambda_0})_{\varphi_0,\lambda}$ satisfies $ww^\star = p$ and $w^\star w = 1$. Conjugating with w, we conclude that the actions $\Lambda_i \curvearrowright P_i^{\Lambda_i}$ are conjugate through a state-preserving isomorphism. $\square$

We are now also able to prove the following weaker version of Theorem E. We say that a sequence of elements g_n in a group Λ is central if for every $h \in \Lambda$, we have that $g_n h = h g_n$ eventually. We call such a sequence trivial if $g_n = e$ eventually. As we see below, the assumption on central sequences in the next theorem exactly ensures that the crossed product $P^\Lambda \rtimes (\Lambda \times \Lambda)$ is full. Note that in Chapter 4, where we prove Theorem E in full generality, we do not need the condition that Λ has no nontrivial central sequences.

Theorem 3.11. *The set of factors*

$$\left\{ (P,\phi)^\Lambda \rtimes (\Lambda \times \Lambda) \;\middle|\; \begin{array}{l} P \text{ a nontrivial amenable factor, } \phi \text{ a normal faithful} \\ \text{almost periodic state, and } \Lambda \text{ an icc group in the} \\ \text{class } \mathcal{C} \text{ without nontrivial central sequences} \end{array} \right\}$$

is exactly classified, up to isomorphism, by the group Λ and the pair (P,ϕ) up to state-preserving isomorphism.

Proof. For $i = 0,1$, let Λ_i be icc groups in the class $\mathcal{C}$ without nontrivial central sequences. Let (P_i,ϕ_i) be amenable factors with a normal faithful almost periodic state. Obviously, if $\Lambda_0 \cong \Lambda_1$ and $(P_0,\phi_0) \cong (P_1,\phi_1)$ through a state-preserving isomorphism, then the von Neumann algebras $P_i^{\Lambda_i} \rtimes (\Lambda_i \times \Lambda_i)$ are isomorphic. Assume conversely that the $P_i^{\Lambda_i} \rtimes (\Lambda_i \times \Lambda_i)$ are isomorphic.

Since Λ_i has trivial center, every nontrivial element of $\Lambda_i \times \Lambda_i$ moves infinitely many elements of Λ_i. By Lemma 2.30, the Bernoulli action of $\Lambda_i \times \Lambda_i$ on $(P_i,\phi_i)^{\Lambda_i}$ is outer. By Lemma 2.26, the almost periodic factor $(P_i,\phi_i)^{\Lambda_i}$ has a factorial discrete decomposition. Finally, since Λ_i has no nontrivial central sequences, there exists a finite subset $\mathcal{F}_i \subset \Lambda_i$ whose centralizer is trivial. Then, the stabilizer of $\{e\} \cup \mathcal{F}_i$ under the action of $\Lambda_i \times \Lambda_i$ on Λ_i is trivial. Thus, by Lemma 2.34, the crossed products $P_i^{\Lambda_i} \rtimes (\Lambda_i \times \Lambda_i)$ are full.

Proceeding exactly as in the proof of Theorem 3.10, using in particular Corollary A.3 and the weak mixing of $\Lambda_1 \times \Lambda_1 \curvearrowright P_1^{\Lambda_1}$, we find an isomorphism $\delta : \Lambda_0 \times \Lambda_0 \to \Lambda_1 \times \Lambda_1$ and a state-preserving isomorphism $\psi : P_0^{\Lambda_0} \to P_1^{\Lambda_1}$ satisfying $\psi \circ \alpha_g^0 = \alpha_{\delta(g)}^1 \circ \psi$ for all $g \in \Lambda_0 \times \Lambda_0$. Here, α^i denotes the Bernoulli action.

We continue with an argument from [PV06, Proof of Theorem 5.4]. Write $\Delta_i = \{(g,g) \mid g \in \Lambda_i\}$. If $\Sigma < \Lambda_i \times \Lambda_i$ is a subgroup such that $\Sigma \cdot g$ is infinite for all $g \in \Lambda_i$, then the action $\Sigma \curvearrowright P_i^{\Lambda_i}$ satisfies Lemma 2.10 (ii) and hence the multiples of 1 are the only $(\alpha_g^i)_{g\in\Sigma}$-invariant elements of $P_i^{\Lambda_i}$. Denote by $\pi_e : P_i \to P_i^{\Lambda_i}$ the embedding as the e-th tensor factor. Since every element of $\pi_e(P_0)$ is Δ_0-invariant, it follows that there exists a $g \in \Lambda_1$ such that $\delta(\Delta_0)\cdot g$ is finite. Composing ψ with $\alpha^1_{(e,g)}$ and δ with $\mathrm{Ad}(e,g)$, we may assume that $g = e$. So we find a finite index subgroup $\Delta_0' < \Delta_0$ such that $\delta(\Delta_0') \subset \Delta_1$.

But then, every element of $\psi^{-1}(\pi_e(P_1))$ is Δ_0'-invariant. Since Λ_0 is icc, the sets $\Delta_0 \cdot g$ are infinite for all $g \neq e$. So also $\Delta_0' \cdot g$ is infinite for all $g \neq e$. Therefore, all Δ_0'-invariant elements of $P_0^{\Lambda_0}$ belong to $\pi_e(P_0)$, by Lemma 2.10. We conclude that $\psi^{-1}(\pi_e(P_1)) \subset \pi_e(P_0)$. Since the elements of $\delta^{-1}(\Delta_1)$ leave every element of $\psi^{-1}(\pi_e(P_1))$ fixed, it follows that $\delta^{-1}(\Delta_1) \subset \Delta_0$. In particular, every element of $\psi(\pi_e(P_0))$ is Δ_1-invariant and thus belongs to $\pi_e(P_1)$. We have proved that $\psi(\pi_e(P_0)) = \pi_e(P_1)$. Then also $\delta(\Delta_0) = \Delta_1$ and the theorem is proved. $\square$

Chapter 4

Bernoulli crossed products without almost periodic weights

In Chapter 3 it was shown that if ϕ is an almost periodic state on an amenable factor P, then the factors $(P,\phi)^{\mathbb{F}_n} \rtimes \mathbb{F}_n$ are completely classified up to isomorphism by n and by the subgroup of $\mathbb{R}_0^+$ generated by the point spectrum of the modular operator Δ_ϕ. In this chapter, based on [Ver15], we improve this classification by also including Bernoulli crossed products for which the state ϕ is not almost periodic. In this setting, the Bernoulli crossed product $P^\Lambda \rtimes \Lambda$ is always of type III_1 (see Lemma 2.25 and Lemma 4.1 below), and does not admit any almost periodic weight (see Remark 4.4). Our classification thus yields many non-isomorphic factors of type III_1 without almost periodic weights, see Section 4.5.

The main result of this chapter is Theorem C, providing an optimal classification result for Bernoulli crossed products with general states ϕ on the base algebra. For every nontrivial factor (P,ϕ) equipped with a normal faithful state, we denote by $P_{\phi,\mathrm{ap}} \subset P$ the *almost periodic part* of P, i.e. $P_{\phi,\mathrm{ap}}$ is the subalgebra spanned by the eigenvectors of Δ_ϕ in P. Note that the theorem in particular applies to the case where the states ϕ_i are weakly mixing, i.e. when $(P_i)_{\phi_i,\mathrm{ap}} = \mathbb{C}$. We denote by $\mathcal{C}$ the class of groups defined in Definition 2.18.

Theorem C. *Let (P_0,ϕ_0) and (P_1,ϕ_1) be nontrivial amenable factors equipped with normal faithful states, such that $(P_0)_{\phi_0,ap}$ and $(P_1)_{\phi_1,ap}$ are factors. Let Λ_0 and Λ_1 be icc groups in the class $\mathcal{C}$.*

The algebras $P_0^{\Lambda_0} \rtimes \Lambda_0$ *and* $P_1^{\Lambda_1} \rtimes \Lambda_1$ *are isomorphic if and only if one of the following statements holds.*

(a) *The states* ϕ_0 *and* ϕ_1 *are both tracial, and the actions* $\Lambda_i \curvearrowright (P_i, \phi_i)^{\Lambda_i}$ *are cocycle conjugate, modulo a group isomorphism* $\Lambda_0 \cong \Lambda_1$*.*

(b) *The states* ϕ_0 *and* ϕ_1 *are both nontracial, and there exist projections* $p_i \in (P_i^{\Lambda_i})_{\phi_i^{\Lambda_i}}$ *such that the reduced cocycle actions* $(\Lambda_i \curvearrowright (P_i, \phi_i)^{\Lambda_i})^{p_i}$ *are cocycle conjugate through a state-preserving isomorphism, modulo a group isomorphism* $\Lambda_0 \cong \Lambda_1$*.*

Remark that the actions $\Lambda_i \curvearrowright P_i^{\Lambda_i}$ are outer, and under the conditions of the theorem, the centralizer of $P_i^{\Lambda_i}$ with respect to $\phi_i^{\Lambda_i}$ is a factor. Hence, the reduced cocycle action $(\Lambda_i \curvearrowright (P_i, \phi_i)^{\Lambda_i})^{p_i}$ makes sense, and is well defined up to cocycle conjugation, see Section 2.2.

Note that if the centralizers of $P_i^{\Lambda_i}$ with respect to $\phi_i^{\Lambda_i}$ are trivial, then we automatically get conjugation of the two actions. In particular, Theorem C implies Theorem B, providing the full classification of Bernoulli crossed products built from weakly mixing states, i.e. where the modular operator Δ_ϕ of the state on the base algebra has no invariant finitely dimensional subspaces.

Theorem B. *The set of factors*

$$\left\{(P,\phi)^\Lambda \rtimes \Lambda \;\middle|\; \begin{array}{l} P \text{ a nontrivial amenable factor with a normal faithful weakly} \\ \text{mixing state } \phi, \text{ and } \Lambda \text{ an icc group in the class } \mathcal{C} \end{array} \right\}$$

is exactly classified, up to isomorphism, by the group Λ *and the action* $\Lambda \curvearrowright (P, \phi)^\Lambda$ *up to a state-preserving conjugacy of the action.*

More generally, if the group Λ_i is a direct product of two icc groups in the class $\mathcal{C}$, or if we consider the two-sided Bernoulli action, we can apply Popa's cocycle superrigidity theorems and also get conjugation. In Section 4.4 below, we show how Theorem C implies the result of Theorems D and E.

Theorem D. *The set of factors*

$$\left\{(P,\phi)^\Lambda \rtimes \Lambda \;\middle|\; \begin{array}{l} P \text{ a nontrivial amenable factor with normal faithful state } \phi \\ \text{such that } P_{\phi,ap} \text{ is a factor, and } \Lambda \text{ a direct product of two} \\ \text{icc groups in the class } \mathcal{C} \end{array} \right\}$$

is exactly classified, up to isomorphism, by the group Λ *and the action* $\Lambda \curvearrowright (P, \phi)^\Lambda$ *up to a state-preserving conjugacy of the action.*

Theorem E. *The set of factors*

$$\Big\{(P,\phi)^\Lambda \rtimes (\Lambda \times \Lambda) \;\Big|\; \begin{array}{l} P \textit{ a nontrivial amenable factor with normal faithful} \\ \textit{state } \phi \textit{ such that } P_{\phi,ap} \textit{ is a factor, and } \Lambda \textit{ an icc} \\ \textit{group in the class } \mathcal{C} \end{array} \Big\}$$

is exactly classified, up to isomorphism, by the group Λ and the pair (P,ϕ) up to a state-preserving isomorphism.

The proofs of the above non-isomorphism results all make use of Tomita-Takesaki's modular theory and the continuous core of type III factors. We first use the main results of [PV11, PV12, Ioa12] providing unique crossed product decomposition theorems for factors of the form $R \rtimes \Lambda$, where $\Lambda \curvearrowright R$ is an outer action on the hyperfinite II_1 factor R of a group Λ in class $\mathcal{C}$. We then obtain a cocycle conjugation isomorphism $\psi : P_0^\Lambda \rtimes \mathbb{R} \to P_1^\Lambda \rtimes \mathbb{R}$ between the induced actions $\Lambda \curvearrowright P_i^\Lambda \rtimes \mathbb{R}$ on the continuous cores. Using the spectral gap methods of [Pop06], we obtain mutual intertwining $\psi(L\mathbb{R}) \prec L\mathbb{R}$, $L\mathbb{R} \prec \psi(L\mathbb{R})$, see Lemma 4.5 below. Under the assumption that the almost periodic parts $(P_i)_{\phi_i,\mathrm{ap}}$ are factors, we are able to deduce that the actions $\Lambda \curvearrowright P_i^\Lambda$ must then be cocycle conjugate, up to reductions, see Lemmas 4.8 and 4.10.

Recently, Houdayer and Isono [HI15] introduced intertwining techniques in the general type III setting. While it is tempting to use these interesting techniques to prove our non-isomorphism results, there are two obstructions to do so. Firstly, the main intertwining theorems for group actions on type II_1 factors [PV11, PV12, Ioa12] do currently not have general analogues in the type III setting. Secondly, it is unclear whether a direct type III approach would allow to recover a state-preserving conjugacy, as in the above theorems. After all, in our situation, we do have 'favorite' states, in contrast to the unique prime factorization problem studied in [HI15].

4.1 The induced action on the continuous core

In this section, we study when for a Bernoulli action $\Lambda \curvearrowright P^I$, the associated action on its continuous core $\Lambda \curvearrowright P^I \rtimes \mathbb{R}$ is properly outer.

If (P,ϕ) is a nontrivial factor equipped with a normal faithful state that is not almost periodic, and $\Lambda \curvearrowright I$ is any faithful action, the induced action $\Lambda \curvearrowright P^I \rtimes \mathbb{R}$ on the continuous core of P^I is always properly outer, as stated in the next result.

Lemma 4.1. *Let (P,ϕ) be a nontrivial factor equipped with a normal faithful not almost periodic state, and let I be a countable set. Let $\alpha : I \to I$ be* *any nontrivial permutation. Denote by* $P^I \rtimes \mathbb{R}$ *the crossed product with the modular action of* ϕ^I*. Then the induced automorphism* $\hat{\alpha} \in \mathrm{Aut}(P^I \rtimes \mathbb{R})$ *on the continuous core, given by*

$$\hat{\alpha}(\pi_k(x)) = \pi_{\alpha(k)}(x), \quad \textit{for } x \in P, k \in I,$$

$$\hat{\alpha}(\lambda(t)) = \lambda(t), \qquad \textit{for } t \in \mathbb{R},$$

is properly outer. Here, $\pi_k : P \to P^I \rtimes \mathbb{R}$ *denotes the embedding at position* k*.*

Proof. Denote $\varphi = \phi^I$, $Q = P_{\phi,\mathrm{ap}}$, $N = P^I \rtimes \mathbb{R}$, and denote by $\tilde{\varphi}$ the dual weight on N with respect to φ. For every finite nonempty subset $\mathcal{F} \subset I$, let $K_{\mathcal{F}} \subset L^2(N, \tilde{\varphi}) = L^2(\mathbb{R}, L^2(P^I, \varphi))$ be the subspace defined as

$$K_{\mathcal{F}} = L^2\big(\mathbb{R}, L^2(Q^I (P \ominus Q)^{\mathcal{F}} Q^I, \varphi)\big).$$

By Lemma 2.28 and the first part of Lemma 2.29, we get the orthogonal decomposition

$$L^2(N \ominus (P^I)_{\varphi,\mathrm{ap}} \rtimes \mathbb{R}, \tilde{\varphi}) = L^2\big(\mathbb{R}, L^2(P^I \ominus (P^I)_{\varphi,\mathrm{ap}}, \varphi)\big) = \bigoplus_{\mathcal{F} \in J} K_{\mathcal{F}},$$

where J denotes the set of all finite nonempty subsets of I. Note that since ϕ is not almost periodic, $K_{\mathcal{F}}$ is nonzero for every $\mathcal{F} \in J$.

Now suppose that $\alpha : I \to I$ is a nontrivial permutation, and denote by $\hat{\alpha} \in \mathrm{Aut}(N)$ the induced automorphism. Assume that $v \in N$ is a unitary such that $vx = \hat{\alpha}(x)v$ for all $x \in N$. In particular, v commutes with $L\mathbb{R}$ and hence $v \in (P^I)_\varphi \mathbin{\overline{\otimes}} L\mathbb{R} \subset Q^I \rtimes \mathbb{R}$ by Lemma 2.8. Observe moreover that $\tilde{\varphi}(vxv^\star) = \tilde{\varphi}(\hat{\alpha}(x)) = \tilde{\varphi}(x)$ for all $x \in N$, hence $v \in N_{\tilde{\varphi}}$. Therefore, the map on $L^2(N, \tilde{\varphi})$ induced by the automorphism $\mathrm{Ad}\, v$ on N is isometric and leaves all subspaces $K_{\mathcal{F}}$ invariant, whereas $\hat{\alpha}$ sends $K_{\mathcal{F}}$ to $K_{\alpha(\mathcal{F})}$ for all finite nonempty subsets $\mathcal{F} \subset I$. This is absurd, as α is a nontrivial permutation. We conclude that $\hat{\alpha}$ is outer. □

If however (P, ϕ) is a nontrivial factor equipped with an almost periodic state, and $\Lambda \curvearrowright I$ is any faithful action, then the induced action $\Lambda \curvearrowright P^I \rtimes \mathbb{R}$ on the continuous core of P^I is not always properly outer. For example, if P is a type I factor and $g \in \Lambda$ moves only a finite number of points of I, i.e. the set $\{k \in I \mid g \cdot k \neq k\}$ is finite, then the induced automorphism by g on $P^I \rtimes \mathbb{R}$ is implemented by a unitary. However, if we require that any nontrivial element moves infinitely many points of I, the induced action $\Lambda \curvearrowright P^I \rtimes \mathbb{R}$ is always properly outer, due to the following result.

Lemma 4.2. *Let (P, ϕ) be a nontrivial factor equipped with a normal faithful almost periodic state, and let I be a countable infinite set. Let $\alpha : I \to I$ be any nontrivial permutation. Denote by $P^I \rtimes \mathbb{R}$ the crossed product with the modular action of ϕ^I. Then the induced automorphism $\hat{\alpha} \in \mathrm{Aut}(P^I \rtimes \mathbb{R})$ on the continuous core, given by*

$$\hat{\alpha}(\pi_k(x)) = \pi_{\alpha(k)}(x), \quad \text{for } x \in P, k \in I,$$

$$\hat{\alpha}(\lambda(t)) = \lambda(t), \qquad \text{for } t \in \mathbb{R},$$

is properly outer, unless P is a type I factor and the set $\{k \in I \mid \alpha(k) \neq k\}$ is finite. Here, $\pi_k : P \to P^I \rtimes \mathbb{R}$ denotes the embedding at position k.

Proof. Denote $\varphi = \phi^I$, $N = P^I \rtimes \mathbb{R}$, and assume that $v \in N$ is a nonzero element such that $vx = \hat{\alpha}(x)v$ for all $x \in N$. Since $L\mathbb{R}$ is fixed under $\hat{\alpha}$, v commutes with $L\mathbb{R}$ and hence $v \in (P^I)_\varphi \mathbin{\overline{\otimes}} L\mathbb{R}$. Putting $Q = (P^I)_\varphi$, we get in particular that the restricted automorphism $\hat{\alpha}|_{Q \overline{\otimes} L\mathbb{R}}$ also satisfies $vx = \hat{\alpha}|_{Q \overline{\otimes} L\mathbb{R}}(x)v$ for all $x \in Q \mathbin{\overline{\otimes}} L\mathbb{R}$. Taking an appropriate $\omega \in (L\mathbb{R})_\star$, we get a nonzero element $w = (1 \otimes \omega)(v) \in Q$ satisfying $wx = \hat{\alpha}(x)w$ for all $x \in Q$. Note that Q is a factor by Lemma 2.26, hence w is a multiple of a unitary, and by normalising, we may assume that w is a unitary.

Denote by Γ the countable subgroup of $\mathbb{R}_0^+$ generated by the point spectrum of Δ_ϕ, endowed with the discrete topology. Put $G = \hat{\Gamma}$ and let $\sigma : G \curvearrowright P^I$ denote the extension of σ^φ to an action of the compact group G, as in Section 3.1.1. By Lemma 3.3, it follows that $\hat{\alpha} = \mathrm{Ad}\, w \circ \sigma_s$ on P^I for some $s \in G$, and by Lemma 2.30, it follows that the set $\{k \in I \mid \alpha(k) \neq k\}$ is then necessarily finite, and P is a type I factor.

For the converse, note that if P is a type I factor and $\alpha : I \to I$ is a permutation such that the set $\{k \in I \mid \alpha(k) \neq k\}$ is finite, then there exists a unitary $u \in P^I$ such that $u\pi_k(x)u^\star = \pi_{\alpha(k)}(x)$ for all $k \in I, x \in P$. Since $\varphi \circ \mathrm{Ad}\, u = \varphi$, we have $u \in (P^I)_{\varphi^I}$, and in particular it follows that $u\lambda(t)u^\star = \lambda(t)$ for all $t \in \mathbb{R}$. We have shown that $\hat{\alpha} = \mathrm{Ad}\, u$ on $P^I \rtimes \mathbb{R}$, thus $\hat{\alpha}$ is not properly outer. □

Lemmas 4.1 and 4.2 combined with Lemma 2.25 now yield a general type classification for all Bernoulli crossed products where the base algebra is a factor, as summarized in the next lemma.

Lemma 4.3. *Let (P, ϕ) be a nontrivial factor equipped with a normal faithful state, and assume that $\Lambda \curvearrowright I$ is a faithful action of a countable group on a countable infinite set I. Assume that either*

(i) *P is not a type* I *factor, or that*

(ii) *for every* $s \in \Lambda - \{e\}$ *with finite conjugacy class, the set* $\{k \in I \mid s \cdot k \neq k\}$ *is either infinite or intersects an infinite orbit.*

Then the Bernoulli crossed product $P^I \rtimes \Lambda$ *is a factor of type*

II_1 *if* ϕ *is tracial,*

$\mathrm{III}_\lambda, \lambda \in (0,1)$ *if* ϕ *is nontracial and periodic with period* $\frac{2\pi}{|\log \lambda|}$, *and*

III_1 *if* ϕ *is not periodic.*

Proof. Throughout the proof, we denote by $P^I \rtimes \mathbb{R}$ and $(P^I \rtimes \Lambda) \rtimes \mathbb{R}$ the crossed products of P^I and $P^I \rtimes \Lambda$ respectively, with respect to the product state ϕ^I on P^I.

Assume first that the state ϕ on P is not almost periodic. Then P cannot be a type I factor, hence $P^I \rtimes \Lambda$ is a factor by Corollary 2.31 (iii), whose continuous core $(P^I \rtimes \Lambda) \rtimes \mathbb{R} = (P^I \rtimes \mathbb{R}) \rtimes \Lambda$ is also a factor, since $P^I \rtimes \mathbb{R}$ is a factor by Lemma 2.25 on which Λ acts outerly by Lemma 4.1. Thus $P^I \rtimes \Lambda$ is a type III_1 factor.

Assume now that the state ϕ on P is almost periodic. Either condition (ii) or (iii) of Corollary 2.31 holds, hence $P^I \rtimes \Lambda$ is a factor. Put $\Lambda_0 < \Lambda$ the subgroup of elements with finite conjugacy classes, and put $\Lambda_1 < \Lambda_0$ the subgroup of elements that do not act properly outerly on P^I, which is either $\{e\}$ when P is not of type I, or equal to the set of elements in $s \in \Lambda_0$ such that $\{k \in I \mid s \cdot k \neq k\}$ is finite if P is of type I, by Lemma 2.30. Since every element $s \in \Lambda_0 - \Lambda_1$ acts properly outerly on $P^I \rtimes \mathbb{R}$ by Lemma 4.2, one gets $\mathcal{Z}\big((P^I \rtimes \mathbb{R}) \rtimes \Lambda\big) \subset (P^I \rtimes \mathbb{R}) \rtimes \Lambda_1$, see Lemma 2.9 and Remark 2.32. Moreover, since for every $s \in \Lambda_1 - \{e\}$ the set $\{k \in I \mid s \cdot k \neq k\}$ is finite, there exists a $k \in I$ such that $s \cdot k \neq k$ and such that $\Lambda \cdot k$ is infinite.

Let ψ be a Λ-invariant state on $P^I \rtimes \mathbb{R}$ given by $\psi(x) = \sum_n \frac{1}{2^{n+1}} \mathrm{Tr}_\varphi(p_n x p_n)$ for projections $p_n \in L\mathbb{R}$ such that $\sum_n p_n = 1$ and $\mathrm{Tr}_\varphi(p_n) = 1$, and denote by $\mathfrak{K}(\mathbb{R})$ the set of compactly supported continuous functions on $\mathbb{R}$, which we identify as a subset of $L\mathbb{R}$ when seen as convolution operators. Let $I_0 \subset I$ be the union of all finite orbits, and put J to be the set of all finite nonempty subsets of I intersecting $I - I_0$ nontrivially. For every $\mathcal{F} \in J$, we denote by $K_{\mathcal{F}}$ the $\|\cdot\|_{2,\psi}$-closure of $(P - \mathbb{C}1)^{\mathcal{F}} \mathfrak{K}(\mathbb{R})$. Then we have the orthogonal decomposition

$$L^2((P^I \rtimes \mathbb{R}) \rtimes \Lambda_0, \psi) \ominus L^2((P^{I_0} \rtimes \mathbb{R}) \rtimes \Lambda_0, \psi) = \bigoplus_{\mathcal{F} \in J} K_{\mathcal{F}} \otimes \ell^2(\Lambda_0).$$

As in the proof of Corollary 2.31, it follows that $L^2((P^I \rtimes \mathbb{R}) \rtimes \Lambda_0, \psi) \ominus L^2((P^{I_0} \rtimes \mathbb{R}) \rtimes \Lambda_0, \psi)$ has no invariant vectors under conjugation by $\lambda(s)$ for $s \in \Lambda$, and thus $\mathcal{Z}\big((P^I \rtimes \mathbb{R}) \rtimes \Lambda\big) \subset (P^{I_0} \rtimes \mathbb{R}) \rtimes \Lambda_0$. But since we also have that

$\mathcal{Z}\big((P^I \rtimes \mathbb{R}) \rtimes \Lambda\big) \subset (P^I \rtimes \mathbb{R}) \rtimes \Lambda_1$, and that for every $s \in \Lambda_1 - \{e\}$ the set $\{k \in I \mid s \cdot k \neq k\}$ intersects $I - I_0$ nontrivially, we get exactly as in the proof of Corollary 2.31 that $\mathcal{Z}\big((P^I \rtimes \mathbb{R}) \rtimes \Lambda\big) \subset \mathcal{Z}(P^I \rtimes \mathbb{R})$.

On the other hand, denoting by $G \subset \mathbb{R}$ the subgroup $G = \{t \in \mathbb{R} \mid \sigma_t^\phi = \mathrm{id}\}$, it follows by Lemma 2.25 that $\mathcal{Z}(P^I \rtimes \mathbb{R}) = L(G) \subset \mathcal{Z}\big((P^I \rtimes \mathbb{R}) \rtimes \Lambda\big)$. We conclude that $\mathcal{Z}\big((P^I \rtimes \Lambda) \rtimes \mathbb{R}\big) = \mathcal{Z}(P^I \rtimes \Lambda)$, and hence $P^I \rtimes \Lambda$ and P^I are of the same type. The result now follows from Lemma 2.25. $\square$

We saw in the previous lemma that if (P, ϕ) is a nontrivial factor equipped with a normal faithful state that is not almost periodic, and Λ is a countable infinite group, then $P^\Lambda \rtimes \Lambda$ is a type III_1 factor. The next remark shows that if Λ is nonamenable, $P^\Lambda \rtimes \Lambda$ does not admit any almost periodic weight.

Remark 4.4. Let (P, ϕ) be a nontrivial factor equipped with a normal faithful state that is not almost periodic, and Λ a countable infinite nonamenable group. By Lemma 2.34, $P^\Lambda \rtimes \Lambda$ is a full factor, and Connes's τ-invariant is the weakest topology on $\mathbb{R}$ making $\mathbb{R} \to \mathrm{Aut}(P) : t \mapsto \sigma_t^\phi$ continuous. Since ϕ is not almost periodic, the completion of this topology cannot be compact. It follows that $P^\Lambda \rtimes \Lambda$ has no almost periodic weights, see the proof of [Con74, Corollary 5.3]. $\diamondsuit$

4.2 A technical lemma

Let (P, ϕ) be a factor equipped with a normal faithful state, and let Λ be a countable group acting on an infinite set I, such that the action $\Lambda \curvearrowright I$ has no invariant mean. Put $\varphi = \phi^I$ and $N = P^I \rtimes_{\sigma_\varphi} \mathbb{R}$. Denote the action $\Lambda \curvearrowright N$, induced by the generalized Bernoulli action of Λ on P^I, by α.

The main result of this section is the following technical lemma, allowing us to locate fixed point subalgebras of N with respect to actions which are outer conjugate to the Bernoulli action α. As in the proof of [BV13, Lemma 3.3], locating such subalgebras allows us to deduce cocycle conjugacies between two Bernoulli actions. The proof of Lemma 4.5 follows the lines of the proof of [Pop06, Theorem 4.1].

Lemma 4.5. *Let $p \in L\mathbb{R} \subset N$ be a projection with $0 < \mathrm{Tr}(p) < \infty$. Assume that $(V_g)_{g \in \Lambda} \in \mathcal{U}(pNp)$ are unitaries, and that $\Omega : \Lambda \times \Lambda \to \mathbb{T}$ is a map such that $V_g \alpha_g(V_h) = \Omega(g,h) V_{gh}$ for all $g, h \in \Lambda$. Assume that $Q \subset pNp$ is a subalgebra such that for all $x \in Q$ and for all $g \in \Lambda$, $V_g \alpha_g(x) V_g^\star = x$, and assume that $q \in Q' \cap pNp$ is a projection such that for all $g \in \Lambda$, $V_g \alpha_g(q) V_g^\star \sim q$ inside $Q' \cap pNp$. Then it follows that $qQ \prec_{pNp} pL\mathbb{R}$.*

Put $M = N \cap (L\mathbb{R})' = (P^I)_\varphi \mathbin{\overline{\otimes}} L\mathbb{R}$. *If it moreover holds that* $q \in Q' \cap pMp$ *and* $qL\mathbb{R} \subset qQ \subset qMq$, *then* $qQ \prec_{pMp} pL\mathbb{R}$.

See Lemma 2.8 for the equality $N \cap (L\mathbb{R})' = (P^I)_\varphi \mathbin{\overline{\otimes}} L\mathbb{R}$.

We use the spectral gap methods of [Pop06], applied to the following variant of Popa's malleable deformation [Pop01], due to Ioana [Ioa06]. To construct this deformation, we quickly recall the free product of von Neumann algebras. Let (M, φ_M) and (N, φ_N) denote two von Neumann algebras equipped with normal faithful states, then the *free product* of (M, φ_M) and (N, φ_N) is the unique von Neumann algebra $M * N$ with normal faithful state φ, generated by a copy of M and a copy of N, such that $\varphi|_M = \varphi_M$, $\varphi|_N = \varphi_N$ and

$$\varphi(x_1 x_2 \cdots x_n) = 0,$$

whenever $n \geq 1$ and the x_i's are elements that alternatingly belong to $M \ominus \mathbb{C}1$ and $N \ominus \mathbb{C}1$.

Let now $\widetilde{P} = P * L\mathbb{Z}$ the free product with respect to the state ϕ and the natural trace on $L\mathbb{Z}$, and denote the induced state also by ϕ. Denote by $u_n, n \in \mathbb{Z}$ the canonical unitaries of $L\mathbb{Z}$. Let $f : \mathbb{T} \to (-\pi, \pi]$ be the unique function determined by $t = \exp(if(t)), t \in \mathbb{T}$, and let $h = f(u_1)$ be the hermitian operator such that $u_1 = \exp(ih)$. Define $u_t = \exp(ith)$ for every $t \in \mathbb{R}$. Equip $\widetilde{P}^I$ with the one-parameter group of state-preserving automorphisms θ_t given by the infinite tensor product of $\operatorname{Ad} u_t$, for $t \in \mathbb{R}$. Define the period 2 automorphism γ of $\widetilde{P}^I$ as the infinite tensor product of the automorphism of $\widetilde{P}$ satisfying $x \mapsto x$ for $x \in P$ and $u_1 \mapsto u_{-1}$.

Put $\widetilde{N} = \widetilde{P}^I \rtimes_{\sigma_\varphi} \mathbb{R}$, and denote also the action $\Lambda \curvearrowright \widetilde{N}$, induced by the generalized Bernoulli action of Λ on P^I, by α. The automorphisms θ_t and γ naturally extend to $\widetilde{N}$, by acting as the identity on $L\mathbb{R}$.

The condition that $\Lambda \curvearrowright I$ has no invariant mean, implies the following result, which is very similar to [BV13, Lemma 3.4].

Lemma 4.6. *Let* $p \in L\mathbb{R} \subset N$ *be a nonzero projection. Assume that* $(V_g)_{g\in\Lambda}$ *are unitaries in* $\mathcal{U}(pNp)$, *and* $\Omega : \Lambda \times \Lambda \to \mathbb{T}$ *is a map such that* $V_g \alpha_g(V_h) = \Omega(g, h) V_{gh}$ *for all* $g, h \in \Lambda$. *The unitary representation*

$$\rho : \Lambda \to \mathcal{U}(L^2(p\widetilde{N}p \ominus pNp, \mathrm{Tr})) : \rho_g(\xi) = V_g \alpha_g(\xi) V_g^\star$$

does not admit almost invariant vectors.

Proof. Note that, replacing V_g by $V_g + (1 - p)$, we may assume that $p = 1$.

Denote by $\tilde{\varphi}$ be the dual weight on $\widetilde{N}$ and N with respect to the weight φ on $\widetilde{P}^I$ and P^I respectively. Define for every finite subset $\mathcal{F} \subset I$, the subspace $K_{\mathcal{F}} \subset L^2(\tilde{N})$ as the $\|\cdot\|_2$-closure of the linear span of $P^I(\widetilde{P} \ominus P)^{\mathcal{F}} P^I \mathfrak{K}(\mathbb{R})$, where $\mathfrak{K}(\mathbb{R})$ denotes the set of compactly supported continuous functions on $\mathbb{R}$, and $\mathfrak{K}(\mathbb{R}) \subset L\mathbb{R}$ as convolution operators. Note that if $x \in P^I(\widetilde{P} \ominus P)^{\mathcal{F}} P^I \mathfrak{K}(\mathbb{R}), y \in P^I(\widetilde{P} \ominus P)^{\mathcal{K}} P^I \mathfrak{K}(\mathbb{R})$ for $\mathcal{F} \neq \mathcal{K}$, then $E_{L\mathbb{R}}(y^\star x) = 0$. In particular, we have the orthogonal decomposition

$$L^2(\widetilde{N} \ominus N, \mathrm{Tr}) = \bigoplus_{\mathcal{F} \in J} K_{\mathcal{F}}.$$

Remark that $\rho_g(K_{\mathcal{F}}) = K_{g\mathcal{F}}$ for all $g \in \Lambda$. Denote by $p_{\mathcal{F}}$ the orthogonal projection onto $K_{\mathcal{F}}$, and put $H = L^2(\widetilde{N} \ominus N, \mathrm{Tr})$.

Assume that ρ does admit almost invariant vectors, then we find an $\mathrm{Ad}\,\rho(\Lambda)$-invariant mean μ on $B(H)$. Let J denote the set of all finite nonempty subsets of I, and define the map $\Theta : \ell^\infty(J) \to B(H)$ given by $\Theta(F) = \sum_{\mathcal{F} \in J} F(\mathcal{F}) p_{\mathcal{F}}$. Since $\Theta(g \cdot F) = \rho_g \Theta(F) \rho_g^\star$, the composition $\mu \circ \Theta$ yields a Λ-invariant mean on J. But then $\Lambda \curvearrowright I$ admits an invariant mean, by Lemma 2.33. This is absurd, hence ρ does not admit almost invariant vectors. $\square$

We will also need the following the following transversality property of (θ_t) in the sense of [Pop06, Lemma 2.1].

Lemma 4.7. *Denote by $E : \widetilde{N} \to N$ the unique trace-preserving conditional expectation, and let $p \in L\mathbb{R}$ be a projection with $0 < \mathrm{Tr}(p) < \infty$. For all $x \in pNp$ and all $t \in \mathbb{R}$, we have the inequality*

$$\|x - \theta_t(x)\|_2 \leq \sqrt{2} \|\theta_t(x) - E(\theta_t(x))\|_2.$$

Proof. Denote by J the set of all finite subsets of I, including the empty set, and put for every $\mathcal{F} \in J$, $K_{\mathcal{F}} \subset L^2(N)$ to be the $\|\cdot\|_2$-closure of the linear span of $(P \ominus \mathbb{C}1)^{\mathcal{F}} \mathfrak{K}(\mathbb{R})$, where $\mathfrak{K}(\mathbb{R})$ denotes the set of all compactly supported continuous functions of $\mathbb{R}$, and let $P_{\mathcal{F}}$ be the projection onto $K_{\mathcal{F}}$ for all $\mathcal{F} \in J$. Denote $\rho_t = |\phi(u_t)|^2$, and note that since E preserves $\tilde{\varphi}$, for all $y \in (P \ominus \mathbb{C}1)\mathfrak{K}(\mathbb{R})$ we have that $E(\theta_t(y)) = E(u_t y u_t^\star) = \phi(u_t) y \phi(u_t^\star) = \rho_t y$. It follows that for all $x \in L^2(pNp)$ and all $t \in \mathbb{R}$,

$$\|E(\theta_t(x))\|_2^2 = \sum_{\mathcal{F} \in J} \|E(\theta_t(P_{\mathcal{F}}(x)))\|_2^2 = \sum_{\mathcal{F} \in J} \rho_t^{2|\mathcal{F}|} \|P_{\mathcal{F}}(x)\|_2^2,$$

$$\begin{aligned}\|x-\theta_t(x)\|_2^2 &= \sum_{\mathcal{F}\in J} \|P_{\mathcal{F}}(x)-\theta_t(P_{\mathcal{F}}(x))\|_2^2 \\ &= \sum_{\mathcal{F}\in J} 2\|P_{\mathcal{F}}(x)\|_2^2 - 2\Re\operatorname{Tr}\big(P_{\mathcal{F}}(x)^\star\theta_t(P_{\mathcal{F}}(x))\big) \\ &= \sum_{\mathcal{F}\in J} 2\|P_{\mathcal{F}}(x)\|_2^2 - 2\Re\operatorname{Tr}\big(P_{\mathcal{F}}(x)^\star E\big(\theta_t(P_{\mathcal{F}}(x))\big)\big) \\ &= \sum_{\mathcal{F}\in J} 2(1-\rho^{|\mathcal{F}|})\|P_{\mathcal{F}}(x)\|_2^2.\end{aligned}$$

Since $\|\theta_t(x)-E(\theta_t(x))\|_2^2 = \|\theta_t(x)\|_2^2-\|E(\theta_t(x))\|_2^2 = \sum_{\mathcal{F}\in J}(1-\rho_t^{2|\mathcal{F}|})\|P_{\mathcal{F}}(x)\|_2^2$, the lemma follows from the fact that $(1-\rho_t^{2|\mathcal{F}|}) \geq (1-\rho_t^{|\mathcal{F}|})$ for all $\mathcal{F}\in J$. $\square$

We are now ready to give a proof of Lemma 4.5. Note that the lemma consists of two statements, we first prove the first statement in several steps.

Proof of the first statement of Lemma 4.5. Let $Q \subset pNp$ a subalgebra such that for all $x \in Q$ and for all $g \in \Lambda$, $V_g\alpha_g(x)V_g^\star = x$, and let $q \in Q' \cap pNp$ be a projection with $V_g\alpha_g(q)V_g^\star \sim_{(Q'\cap pNp)} q$ for all $g \in \Lambda$. We will prove that $qQ \prec_{pNp} pL\mathbb{R}$ in several steps. Define as above the malleable deformation $(\theta_t)_{t\in\mathbb{R}}$ of $\widetilde{N}$, and remark that $\theta_t(p) = p$ for all $t \in \mathbb{R}$.

Step 1. We first prove that there exists a $t_0 > 0$ such that

$$\operatorname{Tr}(y\theta_t(y^\star)) \geq \operatorname{Tr}(q)/2 \quad \text{for all } y \in \mathcal{U}(qQ), 0 \leq t \leq t_0. \tag{4.1}$$

Let ρ be the unitary representation of Λ on $L^2(p(\widetilde{N}\ominus N)p, \operatorname{Tr})$ given by $\rho_g(\xi) = V_g\alpha_g(\xi)V_g^\star$. By Lemma 4.6, we find a constant $\kappa > 0$ and a finite subset $\mathcal{F} \subset \Lambda$ such that

$$\|\xi\|_2 \leq \kappa \sum_{g\in\mathcal{F}} \|\rho_g(\xi)-\xi\|_2 \quad \text{for all } \xi \in L^2(p(\widetilde{N}\ominus N)p, \operatorname{Tr}). \tag{4.2}$$

Put $\epsilon = \|q\|_2/6$ and $\delta = \epsilon/(2\kappa|\mathcal{F}|)$. Take $t_0 > 0$ small enough such that for $0 \leq t \leq t_0$,

$$\|(V_g-\theta_t(V_g))p\|_2 \leq \delta \quad \text{for all } g \in \mathcal{F},$$

$$\|q-\theta_t(q)\|_2 \leq \epsilon.$$

For every $x \in Q$ we have $V_g\alpha_g(x)V_g^\star = x$ for all $g \in \Lambda$, and therefore

$$\|V_g\alpha_g(\theta_t(x))V_g^\star - \theta_t(x)\|_2 \leq 2\delta \quad \text{for all } g \in \mathcal{F} \text{ and all } x \in Q \text{ with } \|x\| \leq 1. \tag{4.3}$$

Denote by $E : \widetilde{N} \to N$ the unique trace-preserving conditional expectation. Whenever $x \in Q$ with $\|x\| \leq 1$, we put $\xi = \theta_t(x) - E(\theta_t(x))$ and conclude from (4.2) and (4.3) that

$$\|\xi\|_2 \leq \kappa \sum_{g \in \mathcal{F}} \|\rho_g(\xi) - \xi\|_2 \leq 2\kappa|\mathcal{F}|\delta = \epsilon.$$

By Lemma 4.7, we conclude that $\|x - \theta_t(x)\|_2 \leq 2\epsilon$ for all $x \in Q$ with $\|x\| \leq 1$, hence $\|y - \theta_t(y)\|_2 \leq 3\epsilon$ for all $y \in qQ$. It follows that for all $y \in \mathcal{U}(qQ)$,

$$|\operatorname{Tr}(y\theta_t(y^\star)) - \operatorname{Tr}(yy^\star)| \leq \|y\|_2\|y - \theta_t(y)\|_2 \leq \|q\|_2 3\epsilon = \operatorname{Tr}(q)/2,$$

and hence $\operatorname{Tr}(y\theta_t(y^\star)) \geq \operatorname{Tr}(q)/2$ for all $y \in \mathcal{U}(qQ)$, proving (4.1).

Step 2. We now prove that there exists a nonzero element $W \in q\widetilde{N}\theta_1(q)$ such that $WW^\star \in Q' \cap q\widetilde{N}q$ and

$$xW = W\theta_1(x) \quad \text{for all } x \in qQ. \tag{4.4}$$

Take $t_0 > 0$ as in Step 1, and fix $t = 2^{-n_0}$ for n_0 a large enough integer such that $t = 2^{-n_0} \leq t_0$. Define $K \subset q\widetilde{N}\theta_1(q)$ as the $\|\cdot\|_2$-closed convex hull of $\{y\theta_t(y^\star) \mid y \in \mathcal{U}(qQ)\}$. By Step 1, the trace of the elements $y\theta_t(y^\star)$ for $y \in \mathcal{U}(qQ)$ is greater than $\operatorname{Tr}(q)/2$. Let now $W \in q\widetilde{N}\theta_t(q)$ be the unique element of minimal $\|\cdot\|_2$-norm in K, then we still have that $\operatorname{Tr}(W) \geq \operatorname{Tr}(q)/2$. Moreover, since K is invariant under the isometric map $u \mapsto xu\theta_t(x^\star)$ for $x \in \mathcal{U}(qQ)$, it follows that $xW = W\theta_t(x)$ for all $x \in Q$.

We will now show how one can iteratively build, for every $n \in \mathbb{N}$, a nonzero element $W^{(n)} \in q\widetilde{N}\theta_{2^n t}(q)$ such that $xW^{(n)} = W^{(n)}\theta_{2^n t}(x)$ for all $x \in Q$. Fix $n \geq 1$, put $s = 2^{n-1}t$ and assume that $W = W^{(n-1)} \in q\widetilde{N}\theta_s(q)$ is a nonzero element satisfying $xW = W\theta_s(x)$ for all $x \in Q$. We will now build $W^{(n)}$. Put for $g \in \Lambda$, $W_g = V_g\alpha_g(W)\theta_s(V_g^\star)$. Since Q is pointwise fixed under $\operatorname{Ad} V_g \circ \alpha_g$, we then also have $xW_g = W_g\theta_s(x)$ for all $x \in Q$ and in particular, $W_gW_g^\star$ commutes with Q.

We claim that we can find some $g \in \Lambda$ such that $\gamma(W^\star)W_g \neq 0$. To prove the claim, denote by q_0 the join of the left support projections of all $W_g, g \in \Lambda$, i.e. the join of the support projections of the elements $W_gW_g^\star = V_g\alpha_g(WW^\star)V_g^\star \in p\widetilde{N}p \cap Q'$. By construction, q_0 is a projection in $p\widetilde{N}p \cap Q'$ that satisfies $q_0 = V_g\alpha_g(q_0)V_g^\star$ for all $g \in \Lambda$, thus it follows from Lemma 4.6 that $q_0 \in N$, and in particular that $\gamma(q_0) = q_0$. Assume now that the claim does not hold, i.e. that for all $g \in \Lambda$, $\gamma(W^\star)W_g = 0$. Then one also has that $\gamma(W^\star)W_gW_g^\star = 0$, and since q_0 is the join of the support projections of $W_gW_g^\star$, it follows that $\gamma(W^\star)q_0 = \gamma(W^\star q_0) = 0$. But note that the right support projection of $W^\star$ equals the left support projection of $W = W_e$, which by construction lies under q_0. In particular, $W^\star q_0 = W^\star$, hence we reached the contradiction that $W^\star = W^\star q_0 = 0$.

Fix now $g \in \Lambda$ such that $\gamma(W^\star)W_g \neq 0$. Note that the nonzero element $\theta_s(\gamma(W^\star)W_g)$ satisfies

$$\begin{aligned} x\theta_s\big(\gamma(W^\star)W_g\big) &= \theta_s\big(\gamma(\theta_s(x)W^\star)W_g\big) \\ &= \theta_s\big(\gamma(W^\star)W_g\theta_s(x)\big) = \theta_s\big(\gamma(W^\star)W_g\big)\theta_{2s}(x), \quad \text{for } x \in Q. \end{aligned}$$

Moreover, the left support of $\theta_s(\gamma(W^\star)W_g)$ lies under q, but the right support only lies under $\theta_{2s}(V_g\alpha_g(q)V_g^\star)$. Since we are given that $V_g\alpha_g(q)V_g^\star$ and q are equivalent projections in $Q' \cap pNp$, we can find a partial isometry $v \in Q' \cap pNp$ such that $vv^\star = V_g\alpha_g(q)V_g^\star$ and $v^\star v = q$. Then the element $W^{(n)} := \theta_s(\gamma(W^\star)W_g)\theta_{2s}(v)$ is by construction nonzero, belongs to $q\widetilde{N}\theta_{2s}(q)$, and satisfies $xW^{(n)} = W^{(n)}\theta_{2s}(x)$ for all $x \in Q$, as desired.

Applying the above procedure n_0 times, one gets a nonzero element $W^{(n_0)} \in q\widetilde{N}\theta_1(q)$ such that $xW^{(n_0)} = W^{(n_0)}\theta_1(x)$ for all $x \in Q$, proving (4.4).

Step 3. For every $\mathcal{F} \subset I$, define $N(\mathcal{F}) = P^{\mathcal{F}} \rtimes_{\sigma_\varphi} \mathbb{R}$. We now show the existence of a finite subset $\mathcal{F} \subset I$ such that $qQ \prec_{pNp} pN(\mathcal{F})p$. Suppose that there is no such finite subset $\mathcal{F} \subset I$. By Theorem 2.11, we then can find for every finite subset $\mathcal{F} \subset I$, every finite subset $\mathcal{K} \subset pNq$ and every $\epsilon > 0$, a unitary $x \in \mathcal{U}(qQ)$ such that

$$\|E_{pN(\mathcal{F})p}(axb^\star)\|_2 < \epsilon \quad \text{for all } a, b \in \mathcal{K}.$$

Note that if $\mathcal{F}, \mathcal{F}' \subset I$ are two finite subsets satisfying $\mathcal{F} \subset \mathcal{F}'$, then $\|E_{pN(\mathcal{F})p}(axb^\star)\|_2 \leq \|E_{pN(\mathcal{F}')p}(axb^\star)\|_2$ for any $a, b \in pNq$, $x \in \mathcal{U}(qQ)$. Hence, applying the above to increasing finite subsets $\mathcal{F}_n$ of I and increasing finite subsets $\mathcal{K}_n$ of a fixed countable dense subset of pNq, one finds a sequence of unitaries $x_n \in \mathcal{U}(qQ)$ such that

$$\|E_{pN(\mathcal{F})p}(ax_nb^\star)\|_2 \to 0 \quad \text{for all } a, b \in pNq \text{ and all finite subsets } \mathcal{F} \subset I.$$

We claim that

$$\|E_{pNp}(a\theta_1(x_n)b^\star)\|_2 \to 0 \quad \text{for all } a, b \in p\widetilde{N}p. \tag{4.5}$$

Since the linear span of all $pN\widetilde{P}^{\mathcal{F}}p$, $\mathcal{F} \subset I$ finite, is $\|\cdot\|_2$-dense in $p\widetilde{N}p$, it suffices to prove (4.5) for all $a, b \in pN\widetilde{P}^{\mathcal{F}}p$ and all finite subsets $\mathcal{F} \subset I$. But for all $a, b \in pN\widetilde{P}^{\mathcal{F}}p$, we have

$$E_{pNp}(a\theta_1(x_n)b^\star) = E_{pNp}\big(a\theta_1(E_{pN(\mathcal{F})p}(x_n))b^\star\big), \tag{4.6}$$

thus the conclusion follows from the choice of the sequence (x_n), so (4.5) is proven. Denote now by $W \in q\widetilde{N}\theta_1(q)$ the nonzero element satisfying (4.4) found in Step 2. Since $x_n \in \mathcal{U}(qQ)$ are unitaries, it follows that

$$\begin{aligned}\|E_{pNp}(WW^\star)\|_2 &= \|E_{pNp}(WW^\star)q\|_2 \\ &= \|E_{pNp}(WW^\star)x_n\|_2 = \|E_{pNp}(W\theta_1(x_n)W^\star)\|_2 \to 0.\end{aligned}$$

As W is nonzero, we reached a contradiction, and we have proven the existence of a finite subset $\mathcal{F} \subset I$ such that $qQ \prec_{pNp} pN(\mathcal{F})p$.

Step 4. For the rest of the proof, we fix for every $g \in \Lambda$, a partial isometry $w_g \in Q' \cap pNp$ such that $w_g^\star w_g = V_g\alpha_g(q)V_g^\star$ and $w_g w_g^\star = q$. In this step, we want to 'upgrade' Step 3 by showing the existence of a finite subset $\mathcal{F} \subset I$, an integer $n \in \mathbb{N}$, a projection $p' \in pN(\mathcal{F})p \otimes M_n(\mathbb{C})$, a unital normal homomorphism $\psi : qQ \to p'\big(pN(\mathcal{F})p \otimes M_n(\mathbb{C})\big)p'$ and a partial isometry $v \in qNp \otimes M_{1,n}(\mathbb{C})$ such that $v^\star v \leq p'$,

$$xv = v\psi(x) \quad \text{for all } x \in qQ, \qquad \text{and } w_g V_g \alpha_g(vv^\star)V_g^\star w_g^\star \not\leq vv^\star \text{ for all } g \in \Lambda. \tag{4.7}$$

This additional condition will turn out to be useful in Step 5 below.

Let $\mathcal{F} \subset I$ denote the finite subset found in Step 3. Since $qQ \prec_{pNp} pN(\mathcal{F})p$, we find $n \in \mathbb{N}$, a projection $p' \in pN(\mathcal{F})p \otimes M_n(\mathbb{C})$, a unital normal homomorphism $\psi : qQ \to p'\big(pN(\mathcal{F})p \otimes M_n(\mathbb{C})\big)p'$ and a partial isometry $v \in qNp \otimes M_{1,n}(\mathbb{C})$ such that $v^\star v \leq p'$ and $xv = v\psi(x)$ for all $x \in qQ$. Put $\mathcal{G} \subset \mathcal{U}(Q' \cap pNp)$ the subgroup of all unitaries $u \in Q' \cap pNp$ satisfying $uq = qu$. Let $\mathcal{K} \subset \Lambda$ and $\mathcal{V} \subset \mathcal{G}$ be finite sets such that

$$\mathrm{Tr}\,\Big(\bigvee_{g \in \mathcal{K}, u \in \mathcal{V}} u w_g V_g \alpha_g(vv^\star)V_g^\star w_g^\star u^\star \Big) > \frac{1}{2}\,\mathrm{Tr}\,\Big(\bigvee_{g \in \Lambda, u \in \mathcal{G}} u w_g V_g \alpha_g(vv^\star)V_g^\star w_g^\star u^\star \Big). \tag{4.8}$$

Remark that for all $g \in \mathcal{K}$, all $u \in \mathcal{V}$, and all $x \in qQ$, we have that $uw_gV_g\alpha_g(v) \neq 0$ and

$$xuw_gV_g\alpha_g(v) = uw_gV_g\alpha_g(v)\,\alpha_g(\psi(x)). \tag{4.9}$$

Put $k = |\mathcal{K}|$, $\ell = |\mathcal{V}|$ and enumerate $\mathcal{K} = \{g_1, \cdots, g_k\}$ and $\mathcal{V} = \{u_1, \cdots, u_\ell\}$. Denoting by $e_{ij} \in M_k(\mathbb{C})$ the elementary matrices, we define

$$\tilde{\psi} : qQ \to p'\big(pN(\mathcal{K} \cdot \mathcal{F})p \otimes M_n(\mathbb{C})\big)p' \otimes M_k(\mathbb{C}) \otimes M_\ell(\mathbb{C}) :$$

$$\tilde{\psi}(x) = \sum_{i=1}^{k} \alpha_{g_i}(\psi(x)) \otimes e_{ii} \otimes 1,$$

and putting $e_i \in M_{1,k}(\mathbb{C})$ resp. $M_{1,\ell}(\mathbb{C})$ to be the vector with 1 at position i and 0 elsewhere, we define $\tilde{v} \in qNp \otimes M_{1,n}(\mathbb{C}) \otimes M_{1,m}(\mathbb{C})$ to be the polar part of

$$\sum_{i=1}^{k} \sum_{j=1}^{\ell} u_j w_{g_i} V_{g_i} \alpha_{g_i}(v) \otimes e_i \otimes e_j .$$

From (4.9) it directly follows that $x\tilde{v} = \tilde{v}\tilde{\psi}(x)$ for all $x \in qQ$. Moreover, it is easy to check that

$$\tilde{v}\tilde{v}^\star = \bigvee_{g \in \mathcal{K}, u \in \mathcal{V}} u w_g V_g \alpha_g(vv^\star) V_g^\star w_g^\star u^\star ,$$

$$w_h V_h \alpha_h(\tilde{v}\tilde{v}^\star) V_h^\star w_h^\star \leq \bigvee_{g \in \Lambda, u \in \mathcal{G}} u w_g V_g \alpha_g(vv^\star) V_g^\star w_g^\star u^\star \quad \text{for } h \in \Lambda.$$

Since $\mathrm{Tr}(w_h V_h \alpha_h(\tilde{v}\tilde{v}^\star) V_h^\star w_h^\star) = \mathrm{Tr}(\tilde{v}\tilde{v}^\star)$, it follows now from (4.8) that the triple $(\mathcal{K} \cdot \mathcal{F}, \tilde{\psi}, \tilde{v})$ satisfies (4.7).

Step 5. In this last step, we will finally show that $qQ \prec_{pNp} pL\mathbb{R}$, using the result obtained in Step 4. Let $\mathcal{F} \subset I$ be a finite subset, $n \in \mathbb{N}$, $p' \in pN(\mathcal{F})p \otimes M_n(\mathbb{C})$ a projection, $\psi_1 : qQ \to p'(pN(\mathcal{F})p \otimes M_n(\mathbb{C}))p'$ a unital normal homomorphism and $v \in qNp \otimes M_{1,n}(\mathbb{C})$ a partial isometry such that $v^\star v \leq p'$ and such that (4.7) is satisfied (with $\psi = \psi_1$). We may moreover assume that the support projection of $E_{pN(\mathcal{F})p \otimes M_n(\mathbb{C})}(v^\star v)$ equals p'.

To show that $qQ \prec_{pNp} pL\mathbb{R}$, it is enough to show that $\psi_1(qQ) \prec pL\mathbb{R} \otimes M_n(\mathbb{C})$ inside $pNp \otimes M_n(\mathbb{C})$. Indeed, since the support of $E_{pN(\mathcal{F})p \otimes M_n(\mathbb{C})}(v^\star v)$ equals p', the latter intertwining can then be combined with the intertwining given by v to obtain that $qQ \prec_{pNp} pL\mathbb{R}$ (see also [Vae07, Remark 3.8]).

Since the action $\Lambda \curvearrowright I$ has no invariant mean, a fortiori all orbits are infinite and thus there exists some $g \in \Lambda$ such that $g\mathcal{F} \cap \mathcal{F} = \emptyset$ (see e.g. [PV06, Lemma 2.4]). Putting $\psi_2 : qQ \to \alpha_g(p')\big(pN(g\mathcal{F})p \otimes M_n(\mathbb{C})\big)\alpha_g(p')$ given by $\psi_2 = \alpha_g \circ \psi_1$, we get that

$$\psi_1(x)\, v^\star w_g V_g \alpha_g(v) = v^\star w_g V_g \alpha_g(v)\, \psi_2(x) \quad \text{for all } x \in qQ. \tag{4.10}$$

By the second part of (4.7), $v^\star w_g V_g \alpha_g(v) \neq 0$.

Suppose now that $\psi_1(qQ) \nprec_{pNp \otimes M_n(\mathbb{C})} pL\mathbb{R} \otimes M_n(\mathbb{C})$, then we find a sequence $x_n \in qQ$ such that $\psi_1(x_n) \in \mathcal{U}(\psi_1(qQ))$ and

$$\|E_{L\mathbb{R} \otimes M_n(\mathbb{C})}(a\psi_1(x_n)b)\|_2 \to 0 \quad \text{for all } a, b \in pNp \otimes M_n(\mathbb{C}).$$

Note that for all $a, b \in pN(\mathcal{F})p \otimes M_n(\mathbb{C})$ and all $a', b' \in pN(g\mathcal{F})p \otimes M_n(\mathbb{C})$,

$$E_{pN(g\mathcal{F})p\otimes M_n(\mathbb{C})}(a'a\psi_1(x_n)bb') = a'E_{L\mathbb{R}\otimes M_n(\mathbb{C})}\big(a\psi_1(x_n)b\big)b',$$

and by density, it follows that $\|E_{pN(g\mathcal{F})p\otimes M_n(\mathbb{C})}(y\psi_1(x_n)z)\|_2 \to 0$ for all $y, z \in pNp \otimes M_n(\mathbb{C})$. Putting $w = v^\star w_g V_g \alpha_g(v)$, we have that

$$\begin{aligned} 0 \leftarrow \|E_{pN(g\mathcal{F})p\otimes M_n(\mathbb{C})}(w^\star\psi_1(x_n)w)\|_2 &= \|E_{pN(g\mathcal{F})p\otimes M_n(\mathbb{C})}(\psi_2(x_n)\, w^\star w)\|_2 \\ &= \|\psi_2(x_n)\, E_{pN(g\mathcal{F})p\otimes M_n(\mathbb{C})}(w^\star w)\|_2 \\ &\geq \|w\psi_2(x_n)\, E_{pN(g\mathcal{F})p\otimes M_n(\mathbb{C})}(w^\star w)\|_2 \\ &= \|\psi_1(x_n)w\, E_{pN(g\mathcal{F})p\otimes M_n(\mathbb{C})}(w^\star w)\|_2 \\ &= \|w\, E_{pN(g\mathcal{F})p\otimes M_n(\mathbb{C})}(w^\star w)\|_2 \\ &\geq \|w^\star w\, E_{pN(g\mathcal{F})p\otimes M_n(\mathbb{C})}(w^\star w)\|_2. \end{aligned}$$

But this is absurd, as $w^\star w$ is nonzero. We conclude that $\psi_1(qQ) \prec pL\mathbb{R} \otimes M_n(\mathbb{C})$ inside $pNp \otimes M_n(\mathbb{C})$, and thus $qQ \prec_{pNp} pL\mathbb{R}$. □

The proof of the second statement is very similar, and therefore, we only summarize how to modify the above proof.

Proof of the second statement of Lemma 4.5. Again consider a subalgebra $Q \subset pNp$ such that for all $x \in Q$ and for all $g \in \Lambda$, $V_g\alpha_g(x)V_g^\star = x$, and a projection $q \in Q' \cap pNp$ with $V_g\alpha_g(q)V_g^\star \sim_{(Q'\cap pNp)} q$ for all $g \in \Lambda$. Assume now moreover that $q \in Q' \cap (L\mathbb{R})' \cap pNp$ and $qL\mathbb{R} \subset qQ \subset qMq$, where $M = N \cap (L\mathbb{R})' = (P^I)_\varphi \mathbin{\overline{\otimes}} L\mathbb{R}$.

As the first two steps in the proof of the first statement are still valid, we obtain by (4.4) a nonzero element $W \in q\widetilde{N}\theta_1(q)$ such that $xW = W\theta_1(x)$ for all $x \in qQ$. Note that, as $qL\mathbb{R} \subset qQ$, we actually have that $W \in q\widetilde{N}\theta_1(q) \cap (L\mathbb{R})'$.

For every $\mathcal{F} \subset I$, define $M(\mathcal{F}) = M \cap N(\mathcal{F}) = (P^{\mathcal{F}})_\varphi \mathbin{\overline{\otimes}} L\mathbb{R}$. We claim that there exists a finite subset $\mathcal{F} \subset I$ such that $qQ \prec_{pMp} pM(\mathcal{F})p$. If the claim fails, then there exists, as in Step 3 above, a sequence of unitaries $x_n \in \mathcal{U}(qQ)$ such that

$$\|E_{pM(\mathcal{F})p}(ax_nb^\star)\|_2 \to 0 \quad \text{for all } a, b \in pMq \text{ and all finite subsets } \mathcal{F} \subset I.$$

By (4.6) and using that $E_{pM(\mathcal{F})p}(x_n) = E_{pN(\mathcal{F})p}(x_n)$, it follows that $\|E_{pNp}(a\theta_1(x_n)b^\star)\|_2 \to 0$ for all $a, b \in p\widetilde{N}p \cap (pL\mathbb{R})'$, and thus $\|E_{pNp}(WW^\star)\|_2 \to 0$, contradiction. Hence the claim is proven.

Replacing N by M and $N(\mathcal{F})$ by $M(\mathcal{F})$ in Steps 4 and 5 above, we then obtain that $qQ \prec_{pMp} pL\mathbb{R}$. Note that the fixed partial isometries w_g and the unitaries in $\mathcal{G}$ do not necessarily commute with $L\mathbb{R}$, but we instead use the fact that $qL\mathbb{R} \subset qQ$ to establish that $uw_gV_g\alpha_g(q) \in (L\mathbb{R})'$ for all $u \in \mathcal{G}, g \in \Lambda$. $\square$

Using Lemma 4.5, we obtain the following result. Note that this result follows immediately from [Pop02, Theorem A.1] if both centralizers $(P_i^{I_i})_{\phi_i^{I_i}}$ are trivial, since then $L\mathbb{R} \subset P_i^{I_i} \rtimes \mathbb{R}$ is a maximal abelian subalgebra.

Lemma 4.8. *Let (P_0, ϕ_0) and (P_1, ϕ_1) be nontrivial factors equipped with normal faithful nonperiodic states, such that $(P_i)_{\phi_1,ap}$ is a factor. Let Λ be a countable group acting on infinite sets I_0, I_1, such that the actions $\Lambda \curvearrowright I_i$ do not admit invariant means. Denote by $\varphi_i = \phi_i^{I_i}$ the canonical state on $P_i^{I_i}$.*

Assume that $\psi : P_0^{I_0} \rtimes_{\sigma^{\varphi_0}} \mathbb{R} \to P_1^{I_1} \rtimes_{\sigma^{\varphi_1}} \mathbb{R}$ is an isomorphism such that the induced actions $\Lambda \curvearrowright P_i^{I_i} \rtimes_{\sigma^{\varphi_i}} \mathbb{R}$ are cocycle conjugate through ψ. Then there exists a nonzero partial isometry $w \in P_1^{I_1} \rtimes_{\sigma^{\varphi_1}} \mathbb{R}$ such that $ww^\star \in \psi(L\mathbb{R})'$, $w^\star w \in (L\mathbb{R})'$ and $\psi(L\mathbb{R})w = wL\mathbb{R}$.

Proof. Let ψ be an isomorphism as in the statement. Put $N_i = P_i^{I_i} \rtimes \mathbb{R}$ and denote the action of Λ on N_i by α^i. Identify $\mathbb{R}_0^+$ as the dual of $\mathbb{R}$ using the pairing $\langle t, \mu\rangle = \mu^{\mathbf{i}t}$ for $t \in \mathbb{R}, \mu \in \mathbb{R}_0^+$, and denote by $\hat{\sigma}^{\varphi_i} : \mathbb{R}_0^+ \curvearrowright N_i$ the dual action with respect to σ^{φ_i}. Note that $\hat{\sigma}_\mu^{\varphi_0}(L\mathbb{R}) = L\mathbb{R}$ for all $\mu \in \mathbb{R}_0^+$ and that $\hat{\sigma}^{\varphi_0}$ commutes with the action of Λ on N_0, hence we may replace ψ by $\psi \circ \hat{\sigma}_\mu^{\varphi_0}$ for an appropiate $\mu \in \mathbb{R}_0^+$ and assume that ψ is a trace-preserving isomorphism implementing the cocycle conjugation. This will allow us to ease the notations.

Step 1. First, we prove that for every nonzero finite trace projection $q \in L\mathbb{R}$, there exists a nonzero partial isometry $w \in \psi(q)N_1q$ such that $ww^\star \in \psi(qL\mathbb{R})'$, $w^\star w \in (qL\mathbb{R})'$ and $\psi(L\mathbb{R})w \subset wL\mathbb{R}$.

Note that N_1 is a factor by Lemma 2.25, since ϕ_1 is nonperiodic. Let $q \in L\mathbb{R}$ be any nonzero finite trace projection, and take $u \in \mathcal{U}(N_1)$ such that $u\psi(q)u^\star = q$. Then $\operatorname{Ad} u \circ \psi$ also defines a cocycle conjugation between the actions $\Lambda \curvearrowright qN_iq$, hence we find a 1-cocycle $V_g, g \in \Lambda$ for α^1 such that

$$(\operatorname{Ad} u \circ \psi) \circ \alpha_g^0 = \operatorname{Ad} V_g \circ \alpha_g^1 \circ (\operatorname{Ad} u \circ \psi), \quad \text{for all } g \in \Lambda.$$

Denote $\psi_u = \operatorname{Ad} u \circ \psi$. By construction, $\psi_u(qL\mathbb{R})$ is an abelian subalgebra of qN_1q, such that for all $x \in \psi_u(qL\mathbb{R})$ and for all $g \in \Lambda$, we have that $V_g\alpha_g^1(x)V_g^\star = x$. By Lemma 4.5, $\psi_u(qL\mathbb{R}) \prec_{qN_1q} qL\mathbb{R}$.

Note that the relative commutant of $qL\mathbb{R}$ inside qN_iq is given by $(qL\mathbb{R})' \cap qN_iq = q((L\mathbb{R})' \cap N_i)q = (P_i^{I_i})_{\varphi_i} \mathbin{\overline{\otimes}} qL\mathbb{R}$, see Lemma 2.8. Put $Q_i = (P_i^{I_i})_{\varphi_i} \mathbin{\overline{\otimes}}$

$qL\mathbb{R}$, and observe that $Q_i' \cap qN_iq = qL\mathbb{R}$ by the assumption that $(P_i)_{\varphi_i,\mathrm{ap}}$ is a factor, see Lemma 2.29. Since $\psi_u(qL\mathbb{R}) \prec_{qN_1q} qL\mathbb{R}$, it follows from Lemma 2.13 that there exists a partial isometry $w_0 \in qN_1q$ such that $w_0w_0^\star \in \psi_u(qL\mathbb{R})'$, $w_0^\star w_0 \in (qL\mathbb{R})'$ and $\psi_u(qL\mathbb{R})w_0 \subset w_0(qL\mathbb{R})$. Then $w = u^\star w_0$ is the desired partial isometry.

Step 2. Fix now a nonzero finite trace projection $q \in L\mathbb{R}$, and let $w \in \psi(q)N_1q$ be a nonzero partial isometry such that $ww^\star \in \psi(qL\mathbb{R})'$, $w^\star w \in (qL\mathbb{R})'$ and $\psi(L\mathbb{R})w \subset wL\mathbb{R}$. Put $p_0 = \psi^{-1}(ww^\star)$, $p_1 = w^\star w$, and put $M_i = p_i(P_i^{I_i} \rtimes \mathbb{R})p_i$. Denote by $\theta : M_0 \to M_1$ the isomorphism given by $\theta = \operatorname{Ad} w^\star \circ \psi$, and note that $\theta(p_0L\mathbb{R}) \subset p_1L\mathbb{R}$, hence, taking the relative commutants,

$$p_0L\mathbb{R} \subset \theta^{-1}(p_1L\mathbb{R}) \subset p_0((P_0^{I_0})_{\varphi_0} \mathbin{\overline{\otimes}} qL\mathbb{R})p_0. \tag{4.11}$$

Let $u \in \mathcal{U}(N_1)$ be a unitary such that $up_1 = w$, and put $R = (\operatorname{Ad} u^\star \circ \psi)^{-1}(qL\mathbb{R})$. Since $\operatorname{Ad} u^\star \circ \psi$ is a cocycle conjugation between the actions $\Lambda \curvearrowright qN_iq$, we find a 1-cocycle $V_g, g \in \Lambda$ for α^0 such that

$$\alpha_g^1 \circ (\operatorname{Ad} u^\star \circ \psi) = (\operatorname{Ad} u^\star \circ \psi) \circ \operatorname{Ad} V_g \circ \alpha_g^0.$$

In particular, for all $x \in R$ we have we have $x = V_g\alpha_g^0(x)V_g^\star$. Furthermore, (4.11) means that $p_0L\mathbb{R} \subset p_0Rp_0 \subset p_0(qN_0q \cap (L\mathbb{R})')p_0$. Also note that for every $g \in \Lambda$, the projections p_1 and $\alpha_g^0(p_1)$ both belong to $(P_1)_{\varphi_1} \mathbin{\overline{\otimes}} qL\mathbb{R}$, and the central traces of these projections inside $(P_1)_{\varphi_1} \mathbin{\overline{\otimes}} qL\mathbb{R}$ coincide. Indeed, $(P_1)_{\varphi_1}$ is a factor by Lemma 2.29 and the central trace is therefore given by the trace-preserving conditional expectation $E : (P_1)_{\varphi_1} \mathbin{\overline{\otimes}} qL\mathbb{R} \to qL\mathbb{R}$, which satisfies $E = E \circ \alpha_g^1$. In particular, the projections p_1 and $\alpha_g^1(p_1)$ are equivalent, and so are p_0, $V_g\alpha_g^0(p_0)V_g^\star$ inside $R' \cap qN_0q$.

By Lemma 4.5, we deduce that $\theta^{-1}(p_1L\mathbb{R}) \prec_{(P_0)_{\varphi_0}\overline{\otimes}qL\mathbb{R}} qL\mathbb{R}$. Since $\theta^{-1}(p_1L\mathbb{R})$ and $qL\mathbb{R}$ are abelian, we then find a partial isometry $v \in (P_0)_{\varphi_0} \mathbin{\overline{\otimes}} qL\mathbb{R}$ with $vv^\star \leq p_0$, such that $\theta^{-1}(p_1L\mathbb{R})v \subset vL\mathbb{R}$. But note that v commutes with $L\mathbb{R}$, hence

$$v^\star vL\mathbb{R} = v^\star(p_0L\mathbb{R})v \subset v^\star\theta^{-1}(p_1L\mathbb{R})v \subset v^\star vL\mathbb{R}.$$

Putting $q_0 = \theta(vv^\star) \leq p_1$, it follows that $q_0 \in \theta(p_0L\mathbb{R})'$, $q_0 \in (p_1L\mathbb{R})'$ and $q_0\theta(p_0L\mathbb{R}) = q_0(p_1L\mathbb{R})$. Then $w_0 = wq_0 \in \psi(q)N_1q$ is a nonzero partial isometry such that $w_0w_0^\star = wq_0w^\star = \psi(vv^\star) \in \psi(qL\mathbb{R})'$, $w_0^\star w_0 = q_0 \in (qL\mathbb{R})'$, and moreover

$$w_0^\star w_0(L\mathbb{R}) = q_0(p_1L\mathbb{R})q_0 = q_0\theta(p_0L\mathbb{R})q_0 = w_0^\star\psi(L\mathbb{R})w_0.$$

We conclude that $\psi(L\mathbb{R})w_0 = w_0L\mathbb{R}$, which completes the proof. □

4.3 A non-isomorphism result for type III Bernoulli crossed products

Recall that for any factor (P,ϕ) equipped with a normal faithful state, $P_{\phi,\mathrm{ap}} \subset P$ denotes the subalgebra spanned by the eigenvectors of Δ_ϕ, i.e.

$$P_{\phi,\mathrm{ap}} = \big(\operatorname{span} \bigcup_{\mu\in\mathbb{R}_0^+} P_{\phi,\mu}\big)''.$$

The following theorem provides a non-isomorphism result for all generalized Bernoulli crossed products, with amenable factors (P,ϕ) as base algebras, for which the almost periodic part $P_{\phi,\mathrm{ap}}$ is again a factor. In particular, it will provide a proof of Theorem C.

Theorem 4.9. *Let (P_0,ϕ_0) and (P_1,ϕ_1) be nontrivial amenable factors equipped with normal faithful states, such that $(P_0)_{\phi_0,ap}$ and $(P_1)_{\phi_1,ap}$ are factors. Let Λ_0 and Λ_1 be icc groups in the class $\mathcal{C}$ that act on infinite sets I_0 and I_1 respectively. Assume for $i=0,1$ that the action $\Lambda_i \curvearrowright I_i$ has no invariant mean, and that for every $g\in\Lambda_i-\{e\}$, the set $\{k\in I_i \mid g\cdot k\neq k\}$ is infinite.*

The algebras $P_0^{I_0}\rtimes\Lambda_0$ and $P_1^{I_1}\rtimes\Lambda_1$ are isomorphic if and only if one of the following statements holds.

(a) *The states ϕ_0 and ϕ_1 are both tracial, and the actions $\Lambda_i\curvearrowright(P_i,\phi_i)^{I_i}$ are cocycle conjugate, modulo a group isomorphism $\Lambda_0\cong\Lambda_1$.*

(b) *The states ϕ_0 and ϕ_1 are both nontracial, and the actions $\Lambda_i\curvearrowright(P_i,\phi_i)^{I_i}$ are, up to reductions and modulo a group isomorphism $\Lambda_0\cong\Lambda_1$, cocycle conjugate through a state-preserving isomorphism.*

Proof. If (a) or (b) holds, then it is clear that the crossed products $P_i^{I_i}\rtimes\Lambda_i$ are isomorphic. For the reverse implication, let $\psi: P_0^{I_0}\rtimes\Lambda_0\to P_1^{I_1}\rtimes\Lambda_1$ be a $\star$-isomorphism. Denote by $\varphi_i=\phi_i^{I_i}$ be the product state on $P_i^{I_i}$. By Lemma 2.25, we may distinguish the following three distinct cases.

Case 1. *One of the states ϕ_i is tracial.*
If one of the states ϕ_i is tracial, $P_0^{I_0}$, $P_0^{I_0}\rtimes\Lambda_0$, $P_1^{I_1}$ and $P_1^{I_1}\rtimes\Lambda_1$ are all necessarily of type II_1.

Since the groups Λ_i are icc and belong to the class $\mathcal{C}$, it follows from Lemma 2.12 that $\psi(P_0^{I_0})$ and $P_1^{I_1}$ are unitarily conjugate inside $P_1^{I_1}\rtimes\Lambda_1$. Since the actions $\Lambda_i\curvearrowright P_i^{I_i}$ are outer, this means that $\Lambda_0\cong\Lambda_1$ and that the actions $\Lambda_i\curvearrowright P_i^{I_i}$ are cocycle conjugate.

Case 2. *One of the states ϕ_i is periodic.*
Note that under the extra assumption that the $P_i^{I_i} \rtimes \Lambda_i$ are full factors, the result directly follows from the proof of Theorem 3.10. We now provide a proof of the general situation. First note that since Λ_i is icc and $\mathcal{Z}(P_i^{I_i} \rtimes \mathbb{R}) \subset L\mathbb{R}$ by Lemma 2.25, we get that $\mathcal{Z}((P_i^{I_i} \rtimes \mathbb{R}) \rtimes \Lambda_i) = \mathcal{Z}(P_i^{I_i} \rtimes \mathbb{R})$, and hence $P_i^{I_i} \rtimes \Lambda_i$ and $P_i^{I_i}$ are of the same type. In particular, it follows from the type classification (see Lemma 2.25) that since one of the states ϕ_i is periodic, also the other is, and that the periods of both states ϕ_i are equal. Let $T > 0$ denote this period.

Put $G = \mathbb{R}/T\mathbb{Z}$, let $\sigma^i : G \curvearrowright P_i^{I_i}$ denote the actions induced by σ^{φ_i}, and put $N_i = P_i^{I_i} \rtimes G$. By Connes's Radon-Nykodym cocycle theorem for modular automorphism groups, ψ is a cocycle conjugacy between the modular actions σ^i of G and therefore extends to a $\star$-isomorphism $\Psi : M_0 \to M_1$ between the crossed products $M_i = (P_i^{I_i} \rtimes \Lambda_i) \rtimes G = N_i \rtimes \Lambda_i$. Note that $P_i^{I_i}$ has a factorial discrete decomposition Lemma 2.26, hence N_i is the hyperfinite II_∞ factor. Also note that since Λ_i has trivial center, the action $\Lambda_i \curvearrowright N_i$ is outer Lemmas 3.4 and 2.30. Proceeding exactly as in the proof of Theorem 3.9, we obtain that the action $\Lambda_0 \curvearrowright (P_0^{I_0}, \varphi_0)$ is cocycle conjugate to the reduced cocycle action $(\Lambda_1 \curvearrowright (P_1^{I_1}, \varphi_1))^p$ for some projection $p \in (P_1^{I_1})_{\varphi_1}$.

Case 3. *Both states ϕ_i are nonperiodic.*
Let $N_i = P_i^{I_i} \rtimes \mathbb{R}$ denote the crossed product of $P_i^{I_i}$ with the modular action of φ_i. Since ϕ_i is not periodic, $P_i^{I_i}$ is a type III_1 factor by Lemma 2.25, and hence N_i is a factor. It follows from either Lemma 4.1 or from Lemma 4.2 that the induced action $\Lambda_i \curvearrowright N_i$ is outer.

By Connes's Radon-Nykodym cocycle theorem for modular automorphism groups, ψ is a cocycle conjugacy between the modular automorphism groups $(\sigma_t^{\varphi_0})_{t\in\mathbb{R}}$ and $(\sigma_t^{\varphi_1})_{t\in\mathbb{R}}$ and therefore extends to a $\star$-isomorphism $\Psi : M_0 \to M_1$ between the crossed products $M_i = (P_i^{I_i} \rtimes \Lambda_i) \rtimes \mathbb{R} = N_i \rtimes \Lambda_i$. Since the actions $\Lambda_i \curvearrowright P_i^{I_i}$ are state-preserving, the action of Λ_i on N_i equals the identity on $L\mathbb{R} \subset N_i$.

We claim that $\Psi(N_0)$ and N_1 are unitarily conjugate inside M_1. Take a projection $p_0 \in L\mathbb{R}$ of finite trace. Then $\Psi(p_0)$ is a projection of finite trace in the II_∞ factor $N_1 \rtimes \Lambda_1$. After a unitary conjugacy of Ψ, we may assume that $\Psi(p_0) = p_1 \in L\mathbb{R}$. Since the projections p_i are Λ_i-invariant, we have

$$p_i M_i p_i = p_i N_i p_i \rtimes \Lambda_i \ .$$

The restriction of Ψ to $p_0 M_0 p_0$ thus yields a $\star$-isomorphism of $p_0 N_0 p_0 \rtimes \Lambda_0$ onto $p_1 N_1 p_1 \rtimes \Lambda_1$. Because the groups Λ_i are icc and belong to the class $\mathcal{C}$, it follows from [IPP05, Lemma 8.4] that $\Psi(p_0 N_0 p_0)$ is unitarily conjugate to $p_1 N_1 p_1$. Since the N_i are II_∞ factors, the claim follows.

By the claim in the previous paragraph, we can choose a unitary $u \in M_1$ such that $u\Psi(N_0)u^\star = N_1$. In particular, $\Lambda_0 \cong \Lambda_1$ and $\operatorname{Ad} u \circ \Psi$ is a cocycle conjugacy between $\Lambda_0 \curvearrowright N_0$ and $\Lambda_1 \curvearrowright N_1$.

Identify $\mathbb{R}_0^+$ as the dual of $\mathbb{R}$ using the pairing $\langle t, \mu\rangle = \mu^{\mathbf{i}t}$ for $t \in \mathbb{R}, \mu \in \mathbb{R}_0^+$, and denote by $\hat{\sigma}^{\varphi_i} : \mathbb{R}_0^+ \curvearrowright N_i \rtimes \Lambda_i$ the dual action with respect to σ^{φ_i}. We will now extend the obtained cocycle conjugacy to a cocycle conjugacy between the actions $\Lambda_i \times \mathbb{R}_0^+ \curvearrowright N_i$. By construction, $\Psi \circ \hat{\sigma}^{\varphi_0}_\mu = \hat{\sigma}^{\varphi_1}_\mu \circ \Psi$ for all $\mu \in \mathbb{R}_0^+$. Therefore, Ψ further extends to a $\star$-isomorphism $\tilde{\Psi} : M_0 \rtimes \mathbb{R}_0^+ \to M_1 \rtimes \mathbb{R}_0^+$ satisfying $\tilde{\Psi}(\lambda(\mu)) = \lambda(\mu)$ for all $\mu \in \mathbb{R}_0^+$. Note that we can view $M_i \rtimes \mathbb{R}_0^+$ as $N_i \rtimes (\Lambda_i \times \mathbb{R}_0^+)$. Putting $\Theta = \operatorname{Ad} u \circ \tilde{\Psi}$, we get an isomorphism $N_0 \rtimes (\Lambda_0 \times \mathbb{R}_0^+) \to N_1 \rtimes (\Lambda_1 \times \mathbb{R}_0^+)$ satisfying

$$\Theta(N_0) = N_1, \quad \Theta(N_0 \rtimes \Lambda_0) = N_1 \rtimes \Lambda_1, \quad \Theta(\lambda(\mu)) = u\hat{\sigma}^{\varphi_1}_\mu(u^\star)\lambda(\mu) \quad \text{for } \mu \in \mathbb{R}_0^+.$$

Using that the actions $\Lambda_i \curvearrowright N_i$ are outer, that elements in $N_i \rtimes \Lambda_i$ have a unique Fourier decomposition, and that $u\hat{\sigma}^{\varphi_1}_\mu(u^\star) \in \mathcal{N}_{N_1 \rtimes \Lambda_1}(N_1)$, this implies that the restriction of Θ to N_0 is a cocycle conjugacy between the actions $\Lambda_i \times \mathbb{R}_0^+ \curvearrowright N_i$, modulo a continuous group homomorphism $\delta : \Lambda_0 \times \mathbb{R}_0^+ \to \Lambda_1 \times \mathbb{R}_0^+$ satisfying $\delta(\Lambda_0) = \Lambda_1$ and $\delta(e, \mu) \in \Lambda_1 \times \{\mu\}$. Since Λ_i has trivial center, this means that $\delta(g, \mu) = (\delta_0(g), \mu)$ for all $g \in \Lambda_0, \mu \in \mathbb{R}_0^+$, and a group isomorphism $\delta_0 : \Lambda_0 \to \Lambda_1$.

Denoting by $\alpha_i : \Lambda_i \curvearrowright I_i$ the given actions, and by $\hat{\alpha}_i : \Lambda_i \to \operatorname{Aut}(P_i^{I_i} \rtimes \mathbb{R})$ the induced actions on the continuous cores, we now have obtained that the actions $\hat{\alpha}_0 \times \hat{\sigma}^{\varphi_0} : \Lambda_0 \times \mathbb{R}_0^+ \curvearrowright N_0$ and $(\hat{\alpha}_1 \circ \delta_0) \times \hat{\sigma}^{\varphi_1} : \Lambda_0 \times \mathbb{R}_0^+ \curvearrowright N_1$ are cocycle conjugate.

Note that for $i = 0, 1$, the centralizer of $P_i^{I_i}$ with respect to ϕ_i is a factor, by Lemma 2.29. By Lemma 4.8 with $\Lambda = \Lambda_0$ acting on I_0 by α_0, and on I_1 by $\alpha_1 \circ \delta_0$, we find a partial isometry $w \in N_1$ such that $ww^\star \in \Theta(L\mathbb{R})'$ and $w^\star w \in (L\mathbb{R})'$, satisfying $\Theta(L\mathbb{R})w = wL\mathbb{R}$. By Lemma 4.10 below, we conclude that a reduction of the action $\alpha^0 : \Lambda_0 \curvearrowright P_0^{I_0}$ is cocycle conjugate to a reduction of $\alpha^1 \circ \delta_0 : \Lambda_0 \curvearrowright P_1^{I_i}$, through a state-preserving isomorphism. □

Lemma 4.10. *Let (P_0, φ_0), (P_1, φ_1) be type III_1 factors with normal faithful states and separable preduals. Let Λ be a countable group, with state-preserving actions $\Lambda \curvearrowright^{\alpha^i} (P_i, \varphi_i)$, such that the centralizers $(P_i)_{\varphi_i}$ are factors. Denote by $P_i \rtimes \mathbb{R}$ the crossed product of P_i with the modular action of φ_i. Let $\Lambda \curvearrowright^{\tilde{\alpha}^i} P_i \rtimes \mathbb{R}$ denote the induced action given by $\tilde{\alpha}^i_s(\pi_{\sigma^{\varphi_i}}(x)) = \pi_{\sigma^{\varphi_i}}(\alpha^i_s(x))$ for $x \in P_i, s \in \Lambda$ and $\tilde{\alpha}^i_s(\lambda(t)) = \lambda(t)$ for $t \in \mathbb{R}$, and let $\mathbb{R}_0^+ \curvearrowright P_i \rtimes \mathbb{R}$ denote the dual action with respect to σ^{φ_i}.*

Assume that $\psi : P_0 \rtimes \mathbb{R} \to P_1 \rtimes \mathbb{R}$ is an isomorphism such that the induced actions $\Lambda \times \mathbb{R}_0^+ \curvearrowright P_i \rtimes \mathbb{R}$ are cocycle conjugate through ψ, and that $w \in P_1 \rtimes \mathbb{R}$ is *a partial isometry such that $ww^\star \in \psi(L\mathbb{R})'$, $w^\star w \in (L\mathbb{R})'$ and $\psi(L\mathbb{R})w = wL\mathbb{R}$. Then the actions $\Lambda \curvearrowright (P_i, \varphi_i)$ are, up to reductions, cocycle conjugate through a state-preserving isomorphism.*

Proof. Put $N_i = P_i \rtimes \mathbb{R}$, and denote by $\psi : N_0 \to N_1$ the given isomorphism and by $w \in N_1$ the partial isometry such that $ww^\star \in \psi(L\mathbb{R})'$, $w^\star w \in (L\mathbb{R})'$ and $\psi(L\mathbb{R})w = wL\mathbb{R}$. Let $V_{g,\mu} \in \mathcal{U}(N_1)$ be a 1-cocycle for the action $\Lambda \times \mathbb{R}_0^+ \curvearrowright N_1$ such that

$$\psi \circ \alpha_g^0 \circ \hat{\sigma}_\mu^{\varphi_0} = \operatorname{Ad} V_{g,\mu} \circ \alpha_g^1 \circ \hat{\sigma}_\mu^{\varphi_1} \circ \psi \qquad \text{for all } g \in \Lambda, \mu \in \mathbb{R}_0^+. \tag{4.12}$$

Identify $\mathbb{R}_0^+$ as the dual of $\mathbb{R}$ using the pairing $\langle t, \mu\rangle = \mu^{\mathbf{i}t}$ for $t \in \mathbb{R}, \mu \in \mathbb{R}_0^+$, and denote by $\hat{\sigma}^{\varphi_i} : \mathbb{R}_0^+ \curvearrowright N_i$ the dual action with respect to σ^{φ_i}. Extend ψ to an isomorphism $\Psi : N_0 \rtimes \mathbb{R}_0^+ \to N_1 \rtimes \mathbb{R}_0^+$ by putting $\Psi(x) = \psi(x)$ for $x \in P_0 \rtimes \mathbb{R}$ and $\Psi(\lambda(\mu)) = V_{1,\mu}\lambda(\mu)$. Put $\kappa \in \mathbb{R}_0^+$ such that $\operatorname{Tr}_{\varphi_1} \circ \psi = \kappa \operatorname{Tr}_{\varphi_0}$, then Ψ scales the dual weights of $\operatorname{Tr}_{\varphi_i}$ by the same factor κ, i.e. $\tilde{\operatorname{Tr}}_{\varphi_1} \circ \tilde{\Psi} = \kappa \tilde{\operatorname{Tr}}_{\varphi_0}$. Identify $N_i \rtimes \mathbb{R}_0^+$ with $P_i \mathbin{\overline{\otimes}} B(L^2(\mathbb{R}_0^+))$ through the isomorphism

$$\Phi_i^{\mathfrak{F}} : N_i \rtimes \mathbb{R}_0^+ \to P_i \mathbin{\overline{\otimes}} B(L^2(\mathbb{R}_0^+)), \quad \Phi_i^{\mathfrak{F}} = \operatorname{Ad}(1 \otimes \mathfrak{F}) \circ \Phi_i,$$

where $\Phi_i : N_i \rtimes \mathbb{R}_0^+ \to P_i \mathbin{\overline{\otimes}} B(L^2(\mathbb{R}))$ denotes the Takesaki duality isomorphism as defined in [Tak03a, Theorem X.2.3], and $\mathfrak{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R}_0^+)$ is the Fourier transform given by

$$\mathfrak{F}(f)(\mu) = \frac{1}{\sqrt{2\pi}} \int_{t\in\mathbb{R}} f(t)\mu^{-\mathbf{i}t} dt, \quad f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}).$$

Note that the duality isomorphism Φ_i maps $\lambda(t) \in L\mathbb{R}$ to $\lambda(t) \in B(L^2(\mathbb{R}))$, and hence $\Phi_i^{\mathfrak{F}}(L\mathbb{R}) = \operatorname{Ad}\mathfrak{F}(L\mathbb{R}) = L^\infty(\mathbb{R}_0^+)$. Also, $x \in (P_i)_{\varphi_i}$ is mapped to $\Phi_i^{\mathfrak{F}}(x) = x \otimes 1$. Let M be the operator affiliated with $L^\infty(\mathbb{R}_0^+)$ given by $M(f)(t) = tf(t)$, and let ω be the weight on $B(L^2(\mathbb{R}_0^+))$ given by $\omega = \operatorname{Tr}(M\cdot)$. Observe that $\tilde{\operatorname{Tr}}_{\varphi_i} = (\varphi_i \otimes \omega) \circ \Phi_i^{\mathfrak{F}}$. Putting $\Theta = \Phi_1^{\mathfrak{F}} \circ \Psi \circ (\Phi_0^{\mathfrak{F}})^{-1}$ and $\hat{w} = \Phi_1^{\mathfrak{F}}(w)$, we get an isomorphism

$$\Theta : P_0 \mathbin{\overline{\otimes}} B(L^2(\mathbb{R}_0^+)) \to P_1 \mathbin{\overline{\otimes}} B(L^2(\mathbb{R}_0^+))$$

satisfying

$$(\varphi_1 \otimes \omega) \circ \Theta = \kappa(\varphi_0 \otimes \omega),$$

$$\Theta \circ (\alpha_g^0 \otimes \mathrm{id}) = \operatorname{Ad}\Phi_1^{\mathfrak{F}}(V_{g,1}) \circ (\alpha_g^1 \otimes \mathrm{id}) \circ \Theta \quad \text{for all } g \in \Lambda,$$

$$\Theta(L^\infty(\mathbb{R}_0^+))\,\hat{w} = \hat{w}\,L^\infty(\mathbb{R}_0^+).$$

Here, the second equality follows from (4.12). Put $p_0 = \Theta^{-1}(\hat{w}\hat{w}^\star)$, $p_1 = \hat{w}^\star\hat{w}$. Taking the relative commutants of the last equality, we obtain that $\Theta(p_0(P_0 \mathbin{\overline{\otimes}} L^\infty(\mathbb{R}_0^+))p_0)\,\hat{w} = \hat{w}\,(p_1(P_1 \mathbin{\overline{\otimes}} L^\infty(\mathbb{R}_0^+))p_1)$. Denote by $M_i = p_i(P_i \mathbin{\overline{\otimes}} L^\infty(\mathbb{R}_0^+))p_i$ and let $\theta : M_0 \to M_1$ be the isomorphism given by $\theta = \operatorname{Ad}\hat{w}^\star \circ \Theta$.

For $i = 0, 1$ and $g \in \Lambda$, note that the projections p_i and $\alpha_g^i(p_i)$ both belong to $(P_i)_{\varphi_i} \mathbin{\overline{\otimes}} L^\infty(\mathbb{R}_0^+)$, since

$$p_i \in \Psi_i^{\mathfrak{F}}(N_i \cap (L\mathbb{R})') = \Psi_i^{\mathfrak{F}}((P_i)_{\varphi_i} \mathbin{\overline{\otimes}} L\mathbb{R}) = (P_i)_{\varphi_i} \mathbin{\overline{\otimes}} L^\infty(\mathbb{R}_0^+).$$

Moreover, the central traces of these projections inside $(P_i)_{\varphi_i} \mathbin{\overline{\otimes}} L^\infty(\mathbb{R}_0^+)$ coincide, since $(P_i)_{\varphi_i}$ is a factor and the central trace is thus given by the trace-preserving conditional expectation $E : (P_i)_{\varphi_i} \mathbin{\overline{\otimes}} L^\infty(\mathbb{R}_0^+) \to L^\infty(\mathbb{R}_0^+)$, which satisfies $E = E \circ \alpha_g^i$.

Therefore, we find a partial isometry $v_{i,g} \in (P_i)_{\varphi_i} \mathbin{\overline{\otimes}} L^\infty(\mathbb{R}_0^+)$ such that $p_i = v_{i,g}v_{i,g}^\star$ and $\alpha_g(p_i) = v_{i,g}^\star v_{i,g}$. Also choose $v_{i,e} = p_i$. Then the formula $\alpha_g^{p_i}(p_ixp_i) = v_{i,g}(\alpha_g^i \otimes \mathrm{id})(p_ixp_i)v_{i,g}^\star$ for $x \in M_i$ defines a cocycle action $\alpha^{p_i} : \Lambda \curvearrowright (p_iM_ip_i, \varphi_i^p)$, where $\varphi_i^{p_i}$ is the state given by $\varphi_i^{p_i}(p_ixp_i) = \varphi_i(p_ixp_i)/\varphi(p_i)$ for $x \in M_i$. Putting $W_g = \hat{w}^\star\Theta(v_{0,g})\Phi_1^{\mathfrak{F}}(V_{g,1})\alpha_g^1(\hat{w})v_{1,g}^\star \in p_1\Phi_1^{\mathfrak{F}}(N_1)p_1 \subset p_1(P_1 \mathbin{\overline{\otimes}} B(L^2\mathbb{R}_0^+))p_1$, we now obtain that, as isomorphisms $M_0 \to M_1$,

$$\theta \circ \alpha_g^{p_0} = \operatorname{Ad}W_g \circ \alpha_g^{p_1} \circ \theta \quad \text{for all } g \in \Lambda. \tag{4.13}$$

Note that by construction, $\alpha_g^{p_i}$ is the trivial action on $p_iL^\infty(\mathbb{R}_0^+)$, and $\theta(p_0L^\infty(\mathbb{R}_0^+)) = p_1L^\infty(\mathbb{R}_0^+)$. Thus we have that $W_g \in p_1\Phi_1^{\mathfrak{F}}(N_1)p_1 \cap (p_1L^\infty(\mathbb{R}_0^+))' = p_1((P_1)_{\varphi_1} \mathbin{\overline{\otimes}} L^\infty(\mathbb{R}_0^+))p_1$.

For $i = 0, 1$, consider the integral decomposition (see [Tak03a, Section VIII.4])

$$L^2(P_i, \varphi_i) \mathbin{\overline{\otimes}} L^2(\mathbb{R}_0^+) = \int_{\mathbb{R}_0^+}^{\oplus} L^2(P_i, \varphi_i)ds,$$

$$(P_i \mathbin{\overline{\otimes}} L^\infty(\mathbb{R}_0^+), \varphi_i \otimes \omega) = \int_{\mathbb{R}_0^+}^{\oplus} (P_i, \varphi_i)\,ds.$$

Note that ds satisfies $\int_{\mu_1}^{\mu_2} ds = \int_{\log\mu_1}^{\log\mu_2} e^t d\lambda(t)$, with $d\lambda$ the Lebesgue measure on $\mathbb{R}$. In this disintegration, we can write $p_i = \int_{\mathbb{R}_0^+}^{\oplus} p_i(s)ds$ for a measurable field of projections $s \mapsto p_i(s)$ in P_i. Then we find as disintegration of M_i,

$$M_i = p_i(P_i \mathbin{\overline{\otimes}} L^\infty(\mathbb{R}_0^+))p_i = \int_{\mathbb{R}_0^+}^{\oplus} p_i(s)P_ip_i(s)ds.$$

Also choose measurable fields of partial isometries $s \mapsto v_{i,g}(s)$ in P_i, and $s \mapsto W_g(s)$ in P_1, such that $v_{i,g} = \int_{\mathbb{R}_0^+}^{\oplus} v_{i,g}(s)ds$ and $W_g = \int_{\mathbb{R}_0^+}^{\oplus} W_g(s)ds$.

Fix a countable set of measurable fields $\mathfrak{X} = \{s \mapsto x(s)\}$, corresponding to a countable dense subset of M_0, such that for almost every s, the set $\{x(s)|x \in \mathfrak{X}\}$ is dense in $p_0(s)P_0p_0(s)$, and such that for all $x \in \mathfrak{X}$ and all $g \in \Lambda$, also the measurable field $s \mapsto v_{0,g}(s)\alpha_g^0(x(s))v_{0,g}(s)^\star$ belongs to $\mathfrak{X}$. Since $\theta : M_0 \to M_1$ is an isomorphism and by uniqueness of disintegrations, we find for almost all $s \in \mathbb{R}_0^+$ an isomorphism $\theta_s : (p_0(s)P_0p_0(s), \varphi_0^{p_0(s)}) \to (p_1(s)P_1p_1(s), \varphi_1^{p_1(s)})$, such that

$$\theta\left(\int_{\mathbb{R}_0^+}^{\oplus} x(s)ds\right) = \int_{\mathbb{R}_0^+}^{\oplus} \theta_s(x(s))ds \qquad \text{for all measurable fields } x \in \mathfrak{X}. \tag{4.14}$$

Combining (4.13) and (4.14), we get that for all measurable fields $s \mapsto x(s)$ in $\mathfrak{X}$,

$$\int_{\mathbb{R}_0^+}^{\oplus} \theta_s\big(v_{0,g}(s)\alpha_g^0(x(s))v_{0,g}(s)^\star\big)ds$$

$$= \int_{\mathbb{R}_0^+}^{\oplus} W_g(s)v_{1,g}(s)\alpha_g^1(\theta_s(x(s)))v_{1,g}(s)^\star W_g(s)^\star ds.$$

We then can find a conull subset $S \subset \mathbb{R}_0^+$ such that for all $s \in S$, θ_s is defined; $\{x(s) \mid x \in \mathfrak{X}\}$ is dense in $p_0(s)P_0p_0(s)$; $v_{i,g}(s)$ is a partial isometry with left support $p_i(s)$ and right support $\alpha_g(p_i(s))$, fixed by all $\{\sigma_t^{\varphi_i} \mid t \in \mathbb{Q}\}$; $W_g(s)$ is a unitary in $p_1(s)P_1p_1(s)$ fixed by all $\{\sigma_t^{\varphi_1} \mid t \in \mathbb{Q}\}$; and such that for all $x \in \mathfrak{X}$,

$$\theta_s\big(v_{0,g}(s)\alpha_g^0(x(s))v_{0,g}(s)^\star\big) = W_g(s)v_{1,g}(s)\alpha_g^1(\theta_s(x(s)))v_{1,g}(s)^\star W_g(s)^\star.$$

In particular, it follows for all $s \in S$ and for all $g \in \Lambda$, that $v_{i,g}(s) \in (P_i)_{\varphi_i}$, $W_g(s) \in (P_1)_{\varphi_1}$, and that

$$\theta_s \circ \operatorname{Ad} v_{0,g}(s) \circ \alpha_g^0 = \operatorname{Ad} W_g(s) \circ \operatorname{Ad} v_{1,g}(s) \circ \alpha_g^1 \circ \theta_s.$$

Choosing some $s \in S$ for which $p_0(s)$ is nonzero, we obtain that the actions $\alpha^i : \Lambda \curvearrowright P_i$ are, up to reductions, cocycle conjugate through a state-preserving isomorphism. $\square$

4.4 Proofs of Theorems B to E

Theorems B and C follow now easily from Theorem 4.9. Nevertheless, we give a detailed proof for the convenience of the reader.

Proof of Theorem C. Let Λ_i be icc groups in the class $\mathcal{C}$, and (P_i, ϕ_i) be nontrivial amenable factors with normal faithful states such that $(P_i)_{\phi_i,\mathrm{ap}}$ are factors. Since the class $\mathcal{C}$ does not contain amenable groups, the action $\Lambda_i \curvearrowright \Lambda_i$ has no invariant mean. It is obvious that every nontrivial $g \in \Lambda_i$ moves infinitely many points of Λ_i. The result follows now directly from Theorem 4.9 with $I_i = \Lambda_i$. □

Proof of Theorem B. For $i = 0, 1$, let $\Lambda_i \in \mathcal{C}$ be an icc group, and let (P_i, ϕ_i) be a nontrivial amenable factor with a normal faithful weakly mixing state ϕ_i. Obviously, if $\Lambda_0 \cong \Lambda_1$ and the actions $\Lambda_i \curvearrowright (P_i, \phi_i)^{\Lambda_i}$ are conjugate, then the von Neumann algebras $P_i^{\Lambda_i} \rtimes \Lambda_i$ are isomorphic. Assume conversely that $(P_0, \phi_0)^{\Lambda_0} \rtimes \Lambda_0 \cong (P_1, \phi_1)^{\Lambda_1} \rtimes \Lambda_1$ are isomorphic.

Since ϕ_i is weakly mixing, Δ_{ϕ_i} has no eigenvalues, and hence we have that $(P_i)_{\phi,\mathrm{ap}} = \mathbb{C}$ for $i = 0, 1$. By Lemma 2.29 and putting $\varphi_i = \phi_i^{\Lambda_i}$, we then also get that $(P_i^{\Lambda_i})_{\varphi_i} = \mathbb{C}$. Applying Theorem C, we get that the groups Λ_i are isomorphic, and that the actions $\Lambda_i \curvearrowright P_i^{\Lambda_i}$ are conjugate modulo the isomorphism $\Lambda_0 \cong \Lambda_1$, through a state-preserving isomorphism. □

For the proof of Theorems D and E, we need the notion of a *generalized 1-cocycle.* Let $\alpha : G \to \mathrm{Aut}(M, \varphi)$ be an action of a locally compact group on a von Neumann algebra (M, φ) with an n.s.f. weight φ. A *generalized 1-cocycle* for α with *support projection* $p \in M_\varphi$ is a continuous map $w : G \to M_\varphi$ such that $w_g \in pM_\varphi \alpha_g(p)$ is a partial isometry with $p = w_g w_g^\star$ and $\alpha_g(p) = w_g^\star w_g$, and

$$w_{gh} = \Omega(g, h) w_g \alpha_g(w_h) \quad \text{for all } g, h \in G,$$

where $\Omega(g, h)$ is a scalar 2-cocycle.

Proof of Theorem D. For $i = 0, 1$, let Λ_i be a direct product of two icc groups in the class $\mathcal{C}$, and let (P_i, ϕ_i) be a nontrivial amenable factor with a normal faithful state ϕ_i such that $(P_i)_{\phi_i,\mathrm{ap}}$ is a factor. Assume that $(P_0, \phi_0)^{\Lambda_0} \rtimes \Lambda_0 \cong (P_1, \phi_1)^{\Lambda_1} \rtimes \Lambda_1$ are isomorphic. Putting $\varphi_i = \phi_i^{\Lambda_i}$, we get by Theorem C that there exists projections $p_i \in (P_i^{\Lambda_i})_{\varphi_i}$, such that the reductions of $\Lambda_i \curvearrowright P_i^{\Lambda_i}$ by p_i are cocycle conjugate modulo the isomorphism $\Lambda_0 \cong \Lambda_1$ in a state-preserving way. By amplifying both actions, we may assume that either $p_0 = 1$ or $p_1 = 1$. By interchanging P_0 and P_1 is necessary, we now assume that $p_0 = 1$. Identifying $\Lambda = \Lambda_0 = \Lambda_1$, writing $N_i = P_i^{\Lambda_i}$ and denoting the action $\Lambda \curvearrowright N_i$ by α^i, this now means that there exists a state-preserving isomorphism $\psi : N_0 \to p_1 N_1 p_1$ and a generalized 1-cocycle $(v_g)_{g\in\Lambda} \in (N_1)_{\varphi_1}$ with support projection p_1 for $\alpha^1 : \Lambda \curvearrowright N_1$, such that $\psi \circ \alpha_g^0 = \mathrm{Ad}\, v_g \circ \alpha_g^1 \circ \psi$ for all $g \in \Lambda$.

Put $M = (P_1)^{\Lambda_1}_{\phi,\text{ap}}$ and note that $(N_1)_{\varphi_1} = M_{\varphi_1}$ by Lemma 2.29. In particular, v_g is a generalized 1-cocycle for the action $\Lambda \curvearrowright M$. Since the action $\Lambda \curvearrowright P_0^{\Lambda_0}$ has no nontrivial globally invariant subspaces, also $\operatorname{Ad} v_g \circ \alpha^1_g$ on M has no such invariant subspaces, and it follows from Corollary A.3 that $v_g = \chi(g) v^\star \alpha^1_g(v)$ for all $g \in \Lambda$, where $\chi : \Lambda \to \mathbb{T}$ is a character and $v \in M_{\varphi_1,\lambda}$ satisfies $v^\star v = p_1$ and $vv^\star = 1$. Then $\operatorname{Ad} v \circ \psi : N_0 \to N_1$ is a state-preserving isomorphism implementing a conjugation between the actions $\Lambda \curvearrowright P_i^{\Lambda_i}$. □

Proof of Theorem E. Let Λ_i be icc groups in the class $\mathcal{C}$, and (P_i, ϕ_i) be nontrivial amenable factors with normal faithful states such that $(P_i)_{\phi_i,\text{ap}}$ are factors. We only need to show that if $P_i^{\Lambda_i} \rtimes \Lambda_i \times \Lambda_i$ are isomorphic, then the groups Λ_i and the pairs (P_i, ϕ_i) are isomorphic. Assume now that $(P_0, \phi_0)^{\Lambda_0} \rtimes \Lambda_0 \times \Lambda_0 \cong (P_1, \phi_1)^{\Lambda_1} \rtimes \Lambda_1 \times \Lambda_1$ are isomorphic. Applying Theorem 4.9, and proceeding exactly as in the proof of Theorem D, using in particular Corollary A.3 and the weak mixingness of $\Lambda_0 \times \Lambda_0 \curvearrowright P_0^{\Lambda_0}$, we obtain an isomorphism $\delta : \Lambda_0 \times \Lambda_0 \to \Lambda_1 \times \Lambda_1$ and a state-preserving isomorphism $\psi : P_0^{\Lambda_0} \to P_1^{\Lambda_1}$ satisfying $\psi \circ \alpha^0_g = \alpha^1_{\delta(g)} \circ \psi$ for all $g \in \Lambda_0 \times \Lambda_0$. Here we denoted the Bernoulli action $\Lambda_i \times \Lambda_i \curvearrowright P_i^{\Lambda_i}$ by α^i.

As in the proof of Theorem 3.11, we continue with an argument from [PV06, Proof of Theorem 5.4]. Write $\Delta_i = \{(g,g) \mid g \in \Lambda_i\}$. If $\Sigma < \Lambda_i \times \Lambda_i$ is a subgroup such that $\Sigma \cdot g$ is infinite for all $g \in \Lambda_i$, then the action $\Sigma \curvearrowright P_i^{\Lambda_i}$ satisfies Lemma 2.10 (ii) and hence the multiples of 1 are the only $(\alpha^i_g)_{g\in\Sigma}$-invariant elements of $P_i^{\Lambda_i}$. Denote by $\pi_e : P_i \to P_i^{\Lambda_i}$ the embedding as the e-th tensor factor. Since every element of $\pi_e(P_0)$ is Δ_0-invariant, it follows that there exists a $g \in \Lambda_1$ such that $\delta(\Delta_0) \cdot g$ is finite. Composing ψ with $\alpha^1_{(e,g)}$ and δ with $\operatorname{Ad}(e,g)$, we may assume that $g = e$. So we find a finite index subgroup $\Delta_0' < \Delta_0$ such that $\delta(\Delta_0') \subset \Delta_1$.

But then, every element of $\psi^{-1}(\pi_e(P_1))$ is Δ_0'-invariant. Since Λ_0 is icc, the sets $\Delta_0 \cdot g$ are infinite for all $g \neq e$. So also $\Delta_0' \cdot g$ is infinite for all $g \neq e$. Therefore, all Δ_0'-invariant elements of $P_0^{\Lambda_0}$ belong to $\pi_e(P_0)$, by Lemma 2.10. We conclude that $\psi^{-1}(\pi_e(P_1)) \subset \pi_e(P_0)$. Since the elements of $\delta^{-1}(\Delta_1)$ leave every element of $\psi^{-1}(\pi_e(P_1))$ fixed, it follows that $\delta^{-1}(\Delta_1) \subset \Delta_0$. In particular, every element of $\psi(\pi_e(P_0))$ is Δ_1-invariant and thus belongs to $\pi_e(P_1)$. We have proved that $\psi(\pi_e(P_0)) = \pi_e(P_1)$. Then also $\delta(\Delta_0) = \Delta_1$ and the theorem is proved. □

4.5 Examples of non-isomorphic Bernoulli crossed products built from weakly mixing states

In this final section, we provide concrete examples of mutually non-isomorphic Bernoulli crossed products built from weakly mixing states. We first recall the construction of the operator algebras arising from Araki-Wyss representations of the canonical anticommutation relations (CAR), as introduced in [AW68, Section 12]. Fix a real Hilbert space K. A (bounded) *representation of the canonical anticommutation relations over* K is a set of linear operators $\alpha(\xi) \in B(H), \xi \in K$ on some Hilbert space H, subject to the following relations:

$$\alpha(\xi)\alpha(\eta) + \alpha(\eta)\alpha(\xi) = 0, \qquad \alpha(\lambda\xi) = \lambda\alpha(\xi), \quad \xi, \eta \in K, \lambda \in \mathbb{R},$$

$$\alpha(\xi)\alpha(\eta)^\star + \alpha(\eta)^\star\alpha(\xi) = \langle \xi, \eta \rangle,$$

Fix now a self-adjoint operator $\rho \in B(K)$ such that[1] $0 \leq \rho \leq 1$ and such that $\operatorname{Ker}\rho = \{0\} = \operatorname{Ker}(1-\rho)$. Put $H = K + iK$ the complexification and consider the *antisymmetric Fock space* $\mathcal{F}_a(H)$ over H, i.e. the space $\mathcal{F}_a(H) = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^\infty \mathcal{F}_a^n(H)$ where Ω is the *vacuum vector* and $\mathcal{F}_a^n(H)$ is the closed subspace of $H^{\otimes n}$ spanned by the vectors

$$\xi_1 \wedge \cdots \wedge \xi_n = \frac{1}{n!} \sum_{\sigma \in S_n} (\operatorname{sgn}\sigma) \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)}, \quad \text{for } \xi_1, \cdots, \xi_n \in H.$$

Let a be the creation operator on $\mathcal{F}_a(H)$ defined by $a(\xi)(\Omega) = \xi$ and $a(\xi)(\xi_1 \wedge \cdots \wedge \xi_n) = \sqrt{n+1}\xi \wedge \xi_1 \wedge \cdots \wedge \xi_n$ for $\xi, \xi_i \in H$ (see [AW64, Appendix B]), then $a(\xi)$ is a representation of the CAR's. For every operator on x on H, denote by $\Gamma(x)$ the operator on $\mathcal{F}_a(H)$ given by $\Gamma(x)(\Omega) = \Omega$ and $\Gamma(x)(\xi_1 \wedge \cdots \wedge \xi_n) = (x\xi_1 \wedge \cdots \wedge x\xi_n)$. Following [AW68, Equation 12.12], we consider the following representations of the CAR's on $\mathcal{F}_a(H) \otimes \mathcal{F}_a(H)$:

$$a_\rho(\eta) = a((1-\rho)^{\frac{1}{2}}\eta) \otimes \mathrm{id} + \Gamma(-1) \otimes a(\rho^{\frac{1}{2}}\eta)^\star, \quad \eta \in K,$$

$$b_\rho(\eta) = \Gamma(-1)a(\rho^{\frac{1}{2}}\eta) \otimes \Gamma(-1) - \mathrm{id} \otimes \Gamma(-1)a((1-\rho)^{\frac{1}{2}}\eta)^\star.$$

Here, $\Gamma(-1)$ is the map on $\mathcal{F}_a(H)$ induced by $\xi \mapsto -\xi$ on H. Put $R(\rho) = R^{\text{left}}(\rho) = \{a_\rho(\eta) \mid \eta \in K\}''$ and $R^{\text{right}} = \{b_\rho(\eta) \mid \eta \in K\}''$, and note that $R^{\text{left}}(\rho)$ and $R^{\text{right}}(\rho)$ commute. From the work of Araki and Woods [AW68, section 12] that $R^{\text{left}}(\rho)$ and $R^{\text{right}}(\rho)$ are amenable factors. Using induction, one can show that $\Omega \otimes \Omega$ is a cyclic vector for $R^{\text{left}}(\rho)$ and $R^{\text{right}}(\rho)$, and since both algebras commute, it is also a separating vector. One can easily compute

[1] On the real Hilbert space K, we call an operator $\rho \in B(K)$ self-adjoint if $\langle \rho\xi, \eta \rangle = \langle \xi, \rho\eta \rangle$ for all $\xi, \eta \in K$, and we write $0 \leq \rho \leq 1$ if $0 \leq \langle \rho\xi, \xi \rangle \leq \langle \xi, \xi \rangle$ for all $\xi \in K$.

that the modular operator Δ_ϕ of the vector state $\phi = \langle \cdot\, \Omega \otimes \Omega, \Omega \otimes \Omega \rangle$ on $R(\rho)$ is given by $\Delta_\phi = \Gamma(\gamma) \otimes \Gamma(\gamma^{-1})$, where γ is the possibly unbounded operator defined by $\gamma = \rho(1-\rho)^{-1}$. In particular, if ρ has a purely continuous spectrum, the same holds for γ, and then Δ_ϕ has trivial point spectrum, thus ϕ is weakly mixing.

For the rest of this section, we fix the following notation. For every symmetric probability measure ν on $\mathbb{R}$, we put $H_\nu = L^2(\mathbb{R}, \nu)$, and denote by $\rho_\nu \in B(H_\nu)$ the multiplication operator

$$(\rho_\nu f)(x) = e^{2\pi x}(e^{2\pi x} + 1)^{-1} f(x), \quad f \in L^2(\mathbb{R}, \nu), x \in \mathbb{R}.$$

Put $\gamma_\nu = \rho_\nu(1 - \rho_\nu)^{-1}$ and note that γ_ν is the multiplication operator $(\gamma_\nu f)(x) = e^{2\pi x} f(x)$. Following [Shl02], we denote by $\tau(\nu)$ the weakest topology on $\mathbb{R}$ making the map $\mathbb{R} \to \mathcal{U}(H_\nu) : t \mapsto \gamma_\nu^{\mathbf{i}t}$ continuous. We denote the CAR algebra associated with ρ_ν by $(R(\nu), \phi_\nu)$. By the above, ϕ_ν is weakly mixing if and only if ν has no atoms.

Recall that Connes's τ-invariant of a full factor M is defined as the weakest topology on $\mathbb{R}$ making the map $\mathbb{R} \to \mathrm{Out}(M) : t \mapsto \sigma_t^\psi$ continuous, where ψ is any normal faithful semifinite weight on M.

Proposition 4.11. *The following statements hold true.*

(1) *Let Λ be a nonamenable group, and let μ be a symmetric probability measure on $\mathbb{R}$. Then Connes's τ-invariant of the full factor $R(\mu)^\Lambda \rtimes \Lambda$ is given by $\tau(\mu)$. In particular, there exists a continuum of symmetric non-atomic measures $\mu_\alpha, \alpha \in I$ on $\mathbb{R}$ so that the Bernoulli crossed products $R(\mu_\alpha)^\Lambda \rtimes \Lambda$ have mutually non-equivalent τ-invariants.*

(2) *Let Λ be any icc group in the class $\mathcal{C}$. There exists a symmetric non-atomic measure μ on $\mathbb{R}$ such that $\tau(R(\mu)^\Lambda \rtimes \Lambda)$ is the usual topology on $\mathbb{R}$, but $R(\mu)^\Lambda \rtimes \Lambda \ncong R(\nu)^\Lambda \rtimes \Lambda$ for any absolutely continuous symmetric probability measure ν on $\mathbb{R}$.*

Proof. Let Λ be a nonamenable group and μ a symmetric probability measure on $\mathbb{R}$. By Lemma 2.34, $\tau(R(\mu)^\Lambda \rtimes \Lambda)$ is the weakest topology on $\mathbb{R}$ making the function $\sigma^{\phi_\mu} : \mathbb{R} \to \mathrm{Aut}(R(\mu)) : t \mapsto \sigma_t^{\phi_\mu}$ continuous. Now note that $\sigma_{t_n}^{\phi_\mu} \to \mathrm{id}$ in $\mathrm{Aut}(R(\mu))$ if and only if $\Delta_{\phi_\mu}^{\mathbf{i}t_n}$ converges strongly to 1, if and only if $\gamma^{\mathbf{i}t_n}$ converges strongly to 1, thus $\tau(R(\mu)^\Lambda \rtimes \Lambda) = \tau(\mu)$. Now (1) follows immediately from [Shl02, Theorem 2.3].

To prove (2), fix $\Lambda \in \mathcal{C}$ an icc group and let μ be a measure on $\mathbb{R}$ such that all its convolution powers μ^{*n}, $n \geq 1$ are singular with respect to the

Lebesgue measure, and such that $\tau(\mu)$ is the usual topology on $\mathbb{R}$, e.g. the measure constructed in [Shl02, Theorem 2.4]. Observe that the infinite tensor product $R(\rho_\mu)^\Lambda$ with respect to ϕ_μ is represented on the Hilbert space $(\mathcal{F}_a(H_\mu) \otimes \mathcal{F}_a(H_\mu))^{\otimes\Lambda}$. We claim that the spectral measure of the operator $\Delta_{\phi_\mu^\Lambda} - P_{\mathbb{C}(\Omega\otimes\Omega)^\Lambda}$ on this Hilbert space is still singular with respect to the Lebesgue measure. It is easy to see that there exists a unitarily equivalence between the Hilbert spaces

$$\big(\mathcal{F}_a(H_\mu) \otimes \mathcal{F}_a(H_\mu)\big)^{\otimes\Lambda} \cong \mathbb{C}\Omega \oplus \bigoplus_{\substack{k+l\geq 1\\ n_i,m_j\in\mathbb{N}_0}} \bigoplus \big(\mathcal{F}_a^{n_1}(H_\mu) \otimes \cdots \otimes \mathcal{F}_a^{n_k}(H_\mu)\big) \otimes \big(\mathcal{F}_a^{m_1}(H_\mu) \otimes \cdots \otimes \mathcal{F}_a^{m_\ell}(H_\mu)\big),$$

such that $(\Omega \otimes \Omega)^\Lambda$ corresponds to Ω and $\Delta_{\phi_\mu}^\Lambda$ to the operator $(\Gamma(\gamma_\mu) \otimes \cdots \otimes \Gamma(\gamma_\mu)) \otimes (\Gamma(\gamma_\mu^{-1}) \otimes \cdots \otimes \Gamma(\gamma_\mu^{-1}))$ on each of the indicated subspaces. By construction, the spectral measure of the latter is absolutely continuous to $\mu^{*n_1} * \cdots * \mu^{*n_k} * (-\mu)^{*m_1} * \cdots * (-\mu)^{*m_\ell}$, hence singular with respect to the Lebesgue measure, proving the claim.

Let now ν be any absolutely continuous probability measure on $\mathbb{R}$. As above, it follows that the spectral measure of $\Delta_{\phi_\nu^\Lambda} - P_{\mathbb{C}(\Omega\otimes\Omega)^\Lambda}$ is still absolutely continuous. Suppose now that $R(\rho_\mu)^\Lambda \rtimes \Lambda \cong R(\rho_\nu)^\Lambda \rtimes \Lambda$, then it follows from Theorem B that there exists an isomorphism $\psi : R(\rho_\mu)^\Lambda \to R(\rho_\nu)^\Lambda$ such that $\phi_\nu^\Lambda \circ \psi = \phi_\mu^\Lambda$, and such that ψ intertwines the Bernoulli actions. In particular, this gives rise to an isomorphism of the underlining Hilbert spaces $\big(\mathcal{F}_a(H_\mu) \otimes \mathcal{F}_a(H_\mu)\big)^{\otimes\Lambda} \to \big(\mathcal{F}_a(H_\nu) \otimes \mathcal{F}_a(H_\nu)\big)^{\otimes\Lambda}$ such that $(\Omega \otimes \Omega)^\Lambda \mapsto (\Omega \otimes \Omega)^\Lambda$, intertwining $\Delta_{\phi_\mu}^{\otimes\Lambda}$ and $\Delta_{\phi_\nu}^{\otimes\Lambda}$. Thus the spectral measures of $\Delta_{\phi_\mu^\Lambda} - P_{\mathbb{C}(\Omega\otimes\Omega)^\Lambda}$ and $\Delta_{\phi_\nu^\Lambda} - P_{\mathbb{C}(\Omega\otimes\Omega)^\Lambda}$ must be absolutely continuous with respect to each other, contradiction. $\square$

Chapter 5

Conclusion

In this thesis we examined when Bernoulli crossed products are isomorphic. We showed that Bernoulli crossed products that are built from a factorial base algebra equipped with an almost periodic state and an icc group belonging to a large class of groups, are completely classified by the isomorphism class of the group and the subgroup of $\mathbb{R}^+$ generated by the point spectrum of the almost periodic state (Theorem A, Chapter 3). We later generalized this classification result to Bernoulli crossed products built from general states on the base factor, provided that the almost periodic part of the base algebra, $P_{\phi,\mathrm{ap}}$, is a factor (Theorem C, Chapter 4). In particular, we obtained a strong classification result for Bernoulli crossed products with a weakly mixing state, i.e. a state for which the modular automorphism group is weakly mixing. For this family, an isomorphism of two crossed products implies a state-preserving conjugation of the Bernoulli actions $\Lambda \curvearrowright (P,\phi)^\Lambda$ (Theorem B). We could also obtain this behaviour for general states, provided that the group considered is a direct product group (Theorem D). Finally, we could also cover an interesting family of generalized Bernoulli crossed products, namely the crossed products one obtains by considering the left-right-shift (Theorem E).

Altogether, our results use Popa's theory of intertwining-by-bimodules to its maximum extent, and provide a complete picture of the Bernoulli crossed products up to isomorphism, reducing the classification problem to classifying the Bernoulli actions itself up to (cocycle) conjugation. However, while our results show different situations where an isomorphism of the Bernoulli crossed products imply that the Bernoulli actions are conjugate in a state-preserving way, it is still unclear when the latter exactly happens. In our work, we could only show the absence of a state preserving conjugation by using invariants of the states. It would be interesting to retrieve a more concrete criterion which describes exactly when two Bernoulli actions are conjugate through a state-preserving isomorphism, and as mentioned in the introduction, the so-called entropy of the state on the base algebra would be a good candidate if the states are almost periodic. The same question for states that are not almost periodic, e.g. weakly mixing, seems even more complex.

While the von Neumann algebras studied in this work come with a ‘canonical’ state that is preserved under the action, there exists different families of type III factors without such a state. To study these families, the intertwining techniques of Popa were recently generalized to the general type III setting by Houdayer and Isono [HI15]. It would certainly be interesting to see whether this can lead to new classification results for e.g. crossed products of actions that do not preserve any state.

Appendix A

Popa's cocycle superrigidity for Connes-Størmer Bernoulli actions

Let (M, φ) be a von Neumann algebra equipped with a normal faithful almost periodic state and an action $(\alpha_g)_{g\in\Lambda}$ of a group Λ by state-preserving automorphisms. A *generalized* 1-*cocycle* for $(\alpha_g)_{g\in\Lambda}$ is a family of partial isometries $w_g \in M_\varphi$ for which there exists a projection $p \in M_\varphi$ such that $w_g w_g^\star = p$, $w_g^\star w_g = \alpha_g(p)$ for all $g \in \Lambda$, and such that

$$w_{gh} = \Omega(g, h) w_g \alpha_g(w_h) \quad \text{for all } \; g, h \in \Lambda \; ,$$

where $\Omega : \Lambda \times \Lambda \to \mathbb{T}$ is a scalar 2-cocycle. We call p the support projection of $(w_g)_{g\in\Lambda}$. Note that $p \in M_\varphi$.

In [Pop01], Sorin Popa proved a cocycle superrigidity theorem for the Connes-Størmer Bernoulli actions of property (T) groups and of w-rigid groups, i.e. groups admitting an infinite normal subgroup with the relative property (T). In [Pop05], cocycle superrigidity was established for the 'classical' (commutative) Bernoulli actions of w-rigid groups with arbitrary countable target groups, or even target groups in Popa's class $\mathcal{U}_{\text{fin}}$. Both in [Pop01, Pop05], the rigidity is provided by Kazhdan's (relative) property (T). In [Pop06], Popa discovered that cocycle superrigidity can also be proved using his spectral gap rigidity, typically for groups Λ that arise as the direct product of an infinite group and a nonamenable group. In [Pop06], only the commutative Bernoulli actions are treated. Mutatis mutandis, the methods of [Pop06] and

[Pop01] can be combined to obtain cocycle superrigidity for Connes-Størmer Bernoulli actions of product groups with general target groups. For the sake of completeness, we include a complete argument in this appendix and benefit from this occasion to state and prove the most general result that can be obtained along these lines. This appendix appeared earlier as an appendix to [VV14].

Theorem A.1. *Let (P,ϕ), (Q,φ) be von Neumann algebras with an almost periodic normal faithful state. Let $\Lambda \curvearrowright^\alpha (Q,\varphi)$ be a state-preserving action. Assume that Λ also acts on the countable set I and consider the (generalized) Bernoulli action $\Lambda \curvearrowright (M,\varphi) = (P,\phi)^I$. We also denote this action by α, as well as the diagonal action*

$$\alpha : \Lambda \curvearrowright M \mathbin{\overline{\otimes}} Q \; .$$

Assume that Λ is generated by the commuting subgroups Γ and Σ satisfying the following two properties: the action $\Gamma \curvearrowright I$ has no invariant mean, and the action $\Sigma \curvearrowright I$ has infinite orbits.

Let $p \in (M \mathbin{\overline{\otimes}} Q)_\varphi$ be a projection and $w_g \in (M \mathbin{\overline{\otimes}} Q)_\varphi$ a generalized 1-cocycle with support projection p. Then, $p = \sum_k p_k$ for projections $p_k \in (M \mathbin{\overline{\otimes}} Q)_\varphi$ such that $p_k w_g = w_g \alpha_g(p_k)$ for all $g \in \Lambda$ and the generalized 1-cocycle $(p_k w_g)_{g\in\Lambda}$ is cohomologous to a generalized 1-cocycle taking values in an amplification of Q_φ.

More precisely, there exist Hilbert spaces H_k, positive numbers $\lambda_k > 0$, projections $q_k \in Q_\varphi \mathbin{\overline{\otimes}} B(H_k)$ with $(\varphi \otimes \mathrm{Tr})(q_k) < \infty$ and elements $v_k \in (M \mathbin{\overline{\otimes}} Q)_{\varphi,\lambda_k} \mathbin{\overline{\otimes}} \overline{H}$ such that

$$v_k v_k^\star = p_k \;\; \text{and} \;\; v_k^\star v_k = 1 \otimes q_k \quad , \quad v_k^\star \, w_g \, \alpha_g(v_k) = 1 \otimes W_{k,g}$$

where $(W_{k,g})_{g\in\Lambda}$ is a generalized 1-cocycle for the (amplified) action α_g on $Q_\varphi \mathbin{\overline{\otimes}} B(H_k)$ with support projection q_k.

In the special case where $Q = \mathbb{C}1$ and the twisted action $\operatorname{Ad} w_g \circ \alpha_g$ is assumed to have a trivial fixed point algebra, much more can be said, see Corollary A.3.

To prove Theorem A.1, we make again use of Ioana's variant [Ioa06] of Popa's malleable deformation [Pop01] of the Connes-Størmer Bernoulli action. We recall from Section 4.2 the construction of this deformation. Consider $(\widetilde{P},\phi) = (P,\phi) * (L\mathbb{Z},\tau)$ and $(\widetilde{M},\varphi) = (\widetilde{P},\phi)^I$. Denote by $(u_n)_{n\in\mathbb{Z}}$ the canonical unitary operators in $L\mathbb{Z}$ and denote by $h \in L\mathbb{Z}$ the selfadjoint element with spectrum $[-\pi,\pi]$ satisfying $u_1 = \exp(ih)$. Define $u_t = \exp(ith)$ for all $t \in \mathbb{R}$. Equip $\widetilde{M}$ with the one-parameter group of state-preserving automorphisms $(\theta_t)_{t\in\mathbb{R}}$ given as the infinite tensor product $(\operatorname{Ad} u_t)^I$. Define the period 2 automorphism γ of

$\widetilde{M}$ as the infinite tensor product of the automorphism of $\widetilde{P}$ satisfying $x \mapsto x$ for all $x \in P$ and $u_1 \mapsto u_{-1}$. Note that $\gamma \circ \theta_t = \theta_{-t} \circ \gamma$. We still denote by θ_t and γ the automorphisms of $\widetilde{M} \mathbin{\overline{\otimes}} Q$ that act as the identity on Q. We need the following variant of [Pop05, Lemma 2.10].

Lemma A.2. *Assume that the action $\Sigma \curvearrowright I$ has infinite orbits.*

1. *If $(w_h)_{h\in\Sigma}$ is a generalized 1-cocycle for the action α_h on $(M \mathbin{\overline{\otimes}} Q)_\varphi$ with support projection p and if $x \in p(\widetilde{M} \mathbin{\overline{\otimes}} Q)p$ satisfies $x = w_h \alpha_h(x) w_h^\star$ for all $h \in \Sigma$, then $x \in M \mathbin{\overline{\otimes}} Q$.*
2. *Assume that H is a Hilbert space, $q_1, q_2 \in Q_\varphi \mathbin{\overline{\otimes}} B(H)$ are projections with $(\varphi \otimes \mathrm{Tr})(q_i) < \infty$ and $(W_{i,h})_{h\in\Sigma}$ are generalized 1-cocycles for the (amplified) action α_h on $Q_\varphi \mathbin{\overline{\otimes}} B(H)$ with support projection q_i and the same scalar 2-cocycle Ω. If $x \in M \mathbin{\overline{\otimes}} q_1(Q \mathbin{\overline{\otimes}} B(H))q_2$ satisfies $x = (1 \otimes W_{1,h})\alpha_h(x)(1 \otimes W_{2,h}^\star)$ for all $h \in \Sigma$, then $x \in 1 \mathbin{\overline{\otimes}} Q \mathbin{\overline{\otimes}} B(H)$.*

Proof. 1. Denote by J the set of all nonempty subsets of I. For every $\mathcal{F} \in J$, we denote by $K_\mathcal{F}$ the $\|\cdot\|_\varphi$-closed linear span of $M(\widetilde{P} \ominus P)^\mathcal{F} M \otimes Q$ inside $L^2(\widetilde{M} \mathbin{\overline{\otimes}} Q)$. Note that $L^2((\widetilde{M} \ominus M) \mathbin{\overline{\otimes}} Q)$ is the orthogonal direct sum of the closed subspaces $K_\mathcal{F}$, $\mathcal{F} \in J$. Denote by $p_\mathcal{F}$ the orthogonal projection of $L^2(\widetilde{M} \mathbin{\overline{\otimes}} Q)$ onto $K_\mathcal{F}$.

Take $x \in p(\widetilde{M} \mathbin{\overline{\otimes}} Q)p$ satisfying $x = w_h \alpha_h(x) w_h^\star$ for all $h \in \Sigma$. Define $\xi \in \ell^2(J)$ given by $\xi(\mathcal{F}) = \|p_\mathcal{F}(x)\|_\varphi$. Since $K_\mathcal{F}$ is an $(M \mathbin{\overline{\otimes}} Q)$-bimodule, we have for all $h \in \Sigma$ that

$$(h \cdot \xi)(\mathcal{F}) = \|p_{h^{-1}\cdot\mathcal{F}}(x)\|_\varphi = \|p_\mathcal{F}(\alpha_h(x))\|_\varphi$$
$$= \|p_\mathcal{F}(w_h\, \alpha_h(x)\, w_h^\star)\|_\varphi = \|p_\mathcal{F}(x)\|_\varphi = \xi(\mathcal{F}) \; .$$

Since $\Sigma \curvearrowright I$ has infinite orbits, there are no nonzero Σ-invariant vectors in $\ell^2(J)$. So $p_\mathcal{F}(x) = 0$ for all $\mathcal{F} \in J$. This means that $x \in M \mathbin{\overline{\otimes}} Q$.

2. We decompose $L^2((M \ominus \mathbb{C}1) \mathbin{\overline{\otimes}} Q \mathbin{\overline{\otimes}} B(H))$ as the orthogonal direct sum of the subspaces $\mathcal{L}_\mathcal{F}$, $\mathcal{F} \in J$, defined as the $\|\cdot\|_\varphi$-closed linear span of $(P \ominus \mathbb{C}1)^\mathcal{F} \otimes Q \otimes \mathcal{HS}(H)$, where $\mathcal{HS}(H)$ denotes the set of Hilbert-Schmidt operators on H. We then reason in the same way as in 1. □

Proof of Theorem A.1. **Step 1:** we prove the following spectral gap property. There exists a $\kappa > 0$ and $g_1, \ldots, g_n \in \Gamma$ such that for all $\xi \in L^2((\widetilde{M} \ominus M) \mathbin{\overline{\otimes}} Q)$, $h \in \Sigma$ and $\mu_i \in \mathbb{T}$, we have

$$\|\xi\|_\varphi^2 \leq \kappa \sum_{i=1}^n \|\mu_i\, \xi - w_{g_i}\, \alpha_{g_i}(\xi)\, \alpha_h(w_{g_i}^\star)\|_\varphi^2 \; . \tag{A.1}$$

Denote by J the set of all nonempty subsets of I. By Lemma 2.33, we can take a $\kappa > 0$ and $g_1, \ldots, g_n \in \Gamma$ such that

$$\|\eta\|_2^2 \leq \kappa \sum_{i=1}^{n} \|\eta - g_i \cdot \eta\|_2^2 \quad \text{for all } \eta \in \ell^2(J) \; . \tag{A.2}$$

We choose κ such that also $\kappa \geq 1/n$. To prove (A.1), fix $h \in \Sigma$ and $\xi \in L^2((\widetilde{M} \ominus M) \mathbin{\overline{\otimes}} Q)$. Put $\xi_0 = p\xi\alpha_h(p)$. The left hand side of (A.1) equals $\|\xi - \xi_0\|_\varphi^2 + \|\xi_0\|_\varphi^2$, while the right hand side equals

$$\kappa n \|\xi - \xi_0\|_\varphi^2 + \kappa \sum_{i=1}^{n} \|\mu_i \, \xi_0 - w_{g_i} \, \alpha_{g_i}(\xi_0) \, \alpha_h(w_{g_i}^\star)\|_\varphi^2 \; .$$

Since $\kappa n \geq 1$, we may from the beginning assume that $\xi = \xi_0$, i.e. that $\xi = p\xi\alpha_h(p)$.

As in the proof of Lemma A.2.1, we decompose $L^2((\widetilde{M} \ominus M) \mathbin{\overline{\otimes}} Q)$ as the orthogonal direct sum of the closed subspaces $K_{\mathcal{F}}$, with orthogonal projection $p_{\mathcal{F}}$ onto $K_{\mathcal{F}}$. Define $\eta \in \ell^2(J)$ given by $\eta(\mathcal{F}) = \|p_{\mathcal{F}}(\xi)\|_\varphi$. Note that $\|\xi\|_\varphi = \|\eta\|_2$. Since $K_{\mathcal{F}}$ is an $(M \mathbin{\overline{\otimes}} Q)$-bimodule, and $\xi = p\xi\alpha_h(p)$, we have for all $g \in \Gamma$ that

$$(g \cdot \eta)(\mathcal{F}) = \|p_{g^{-1} \cdot \mathcal{F}}(\xi)\|_\varphi = \|p_{\mathcal{F}}(\alpha_g(\xi))\|_\varphi = \|p_{\mathcal{F}}(w_g \, \alpha_g(\xi) \, \alpha_h(w_g^\star))\|_\varphi \; .$$

Therefore, for all $\mu \in \mathbb{T}$, we have

$$\begin{aligned}
\|\eta - g \cdot \eta\|_2^2 &= \sum_{\mathcal{F} \in J} \big| \, \|p_{\mathcal{F}}(\mu\xi)\|_\varphi - \|p_{\mathcal{F}}(w_g \, \alpha_g(\xi) \, \alpha_h(w_g^\star))\|_\varphi \, \big|^2 \\
&\leq \sum_{\mathcal{F} \in J} \|p_{\mathcal{F}}(\mu\xi - w_g \, \alpha_g(\xi) \, \alpha_h(w_g^\star))\|_\varphi^2 \\
&= \|\mu\xi - w_g \, \alpha_g(\xi) \, \alpha_h(w_g^\star)\|_\varphi^2 \; .
\end{aligned}$$

Taking $\mu = \mu_i$, $g = g_i$ and summing over i, (A.1) follows from (A.2).

Step 2: we prove that there exists a $t_0 > 0$ such that

$$\varphi(\theta_t(w_h) \, w_h^\star) \geq \frac{1}{2} \varphi(p) \quad \text{for all } h \in \Sigma, 0 \leq t \leq t_0 \; . \tag{A.3}$$

Since Σ and Γ commute inside Λ, the formula

$$\kappa(g,h) = \Omega(g,h) \, \overline{\Omega}(h,g)$$

defines a bicharacter $\kappa : \Gamma \times \Sigma \to \mathbb{T}$. The 1-cocycle relation implies that

$$\kappa(g,h) \, w_h = w_g \, \alpha_g(w_h) \, \alpha_h(w_g^\star) \quad \text{for all } h \in \Sigma, g \in \Gamma \; .$$

Applying θ_t, it follows that

$$\|\kappa(g,h)\,\theta_t(w_h) - w_g\,\alpha_g(\theta_t(w_h))\,\alpha_h(w_g^\star)\|_\varphi \leq 2\|w_g - \theta_t(w_g)\|_\varphi \; .$$

We write $\xi_h = \theta_t(w_h) - E_{M\overline{\otimes}Q}(\theta_t(w_h))$ and conclude that

$$\|\kappa(g,h)\,\xi_h - w_g\,\alpha_g(\xi_h)\,\alpha_h(w_g)^\star\|_\varphi \leq 2\|w_g - \theta_t(w_g)\|_\varphi \; .$$

In combination with (A.1), it follows that

$$\|\xi_h\|_\varphi^2 \leq 4\kappa \sum_{i=1}^n \|w_{g_i} - \theta_t(w_{g_i})\|_\varphi^2 \; .$$

Fix $t_0 > 0$ such that the right hand side of the previous expression is smaller than $\varphi(p)/2$ for all $0 \leq t \leq t_0$. It follows that $\|\xi_h\|_\varphi^2 \leq \varphi(p)/2$ and hence

$$\|E_{M\overline{\otimes}Q}(\theta_t(w_h))\|_\varphi^2 \geq \frac{1}{2}\varphi(p) \quad \text{for all} \;\; h \in \Sigma, 0 \leq t \leq t_0 \; . \tag{A.4}$$

Let $x \in M \mathbin{\overline{\otimes}} Q$, and write $x = \sum_\mathcal{F} x_\mathcal{F}$, where $x_\mathcal{F}$ is the orthogonal projection of x onto the $\|\cdot\|_\varphi$-closed linear span of $(P \ominus \mathbb{C})^\mathcal{F} \otimes Q$ inside $L^2(M \mathbin{\overline{\otimes}} Q)$, and $\mathcal{F}$ runs over all finite subsets of I. Denoting $\rho(t) = |\varphi(u_t)|^2$, we have $E_{M\overline{\otimes}Q}(\theta_t(x)) = \sum_\mathcal{F} \rho(t)^{|\mathcal{F}|} x_\mathcal{F}$, and hence

$$\begin{aligned}\varphi(x^\star E_{M\overline{\otimes}Q}(\theta_t(x))) &= \sum_\mathcal{F} \rho(t)^{|\mathcal{F}|}\|x_\mathcal{F}\|_2^2 \\ &\geq \sum_\mathcal{F} \rho(t)^{2|\mathcal{F}|}\|x_\mathcal{F}\|_2^2 = \|E_{M\overline{\otimes}Q}(\theta_t(x))\|_2^2.\end{aligned}$$

We obtain for all $x \in (M \mathbin{\overline{\otimes}} Q)_\varphi$ the following transversality inequality in the sense of [Pop06, Lemma 2.1]:

$$\varphi(\theta_t(x)x^\star) = \varphi(x^\star\theta_t(x)) \geq \|E_{M\overline{\otimes}Q}(\theta_t(x))\|_\varphi^2 \; .$$

In combination with (A.4), we have proved that (A.3) holds.

Step 3: there exists a partial isometry $V \in (\widetilde{M}\overline{\otimes}Q)_\varphi$ and a nonzero projection $p_0 \in (M \mathbin{\overline{\otimes}} Q)_\varphi$ satisfying $p_0 \leq p$, $V^\star V = p_0$, $VV^\star = \theta_1(p_0)$ and

$$V\,w_h = \theta_1(w_h)\,\alpha_h(V) \quad \text{for all} \;\; h \in \Sigma \; . \tag{A.5}$$

Take a positive integer r such that $2^{-r} \leq t_0$. Put $t = 2^{-r}$. Define $K \subset (\widetilde{M} \mathbin{\overline{\otimes}} Q)_\varphi$ as the $\|\cdot\|_2$-closed convex hull of $\{\theta_t(w_h)w_h^\star \mid h \in \Sigma\}$. Define V_1 as the unique element of minimal $\|\cdot\|_2$ in K. By (A.3), we have that $\varphi(V_1) \geq \varphi(p)/2$, so that V_1 is nonzero. For all $h, h' \in \Sigma$, we have

$$\theta_t(w_h)\,\alpha_h(x)\,w_h^\star = y \quad \text{where} \;\; x = \theta_t(w_{h'})w_{h'}^\star \;\; , \;\; y = \theta_t(w_{hh'})w_{hh'}^\star \; .$$

Therefore, K is invariant under $\pi_h : K \to K : \pi_h(x) = \theta_t(w_h)\,\alpha_h(x)\,w_h^\star$. Since $x = \theta_t(p)xp$ for all $x \in K$, the maps π_h are isometric. Therefore, $\pi_h(V_1) = V_1$ for all $h \in \Sigma$. This means that $\theta_t(w_h)\,\alpha_h(V_1) = V_1\,w_h$ for all $h \in \Sigma$. Similarly, K is invariant under $x \mapsto \theta_t(\gamma(x))^\star$, so that $V_1^\star = \theta_t(\gamma(V_1))$.

Taking the polar decomposition of V_1, we may assume that V_1 is a nonzero partial isometry with $V_1^\star V_1 \leq p$ and $V_1 V_1^\star \leq \theta_t(p)$. Since $\alpha_h(V_1^\star V_1) = w_h^\star\,V_1^\star V_1\,w_h$ for all $h \in \Sigma$, Lemma A.2.1 implies that $p_0 := V_1^\star V_1 \in (M \mathbin{\overline{\otimes}} Q)_\varphi$. Then $V_1 V_1^\star = \theta_t(p_0)$ and it follows that $V_2 := \theta_t(V_1\,\gamma(V_1^\star))$ is a partial isometry in $(\widetilde{M} \mathbin{\overline{\otimes}} Q)_\varphi$ with left support $\theta_{2t}(p_0)$ and right support p_0, satisfying $\theta_{2t}(w_h)\,\alpha_h(V_2) = V_2\,w_h$ for all $h \in \Sigma$. Continuing the same procedure up to r steps, we arrive at (A.5).

Step 4: adding the subscript 0 to indicate the linear span of all eigenvectors of the modular automorphism group of φ, we prove that there is no sequence $h_n \in \Sigma$ satisfying

$$\begin{aligned} & h_n \cdot i \to \infty && \text{for all } \ i \in I, \\ \text{and} \quad & \|(\varphi \otimes \mathrm{id})((x^\star \otimes 1)w_{h_n}(\alpha_{h_n}(y) \otimes 1))\|_\varphi \to 0 && \text{for all } \ x, y \in M_0 \ . \end{aligned} \tag{A.6}$$

Assume by contradiction that (h_n) is such a sequence. We claim that

$$\|E_{M \overline{\otimes} Q}((x^\star \otimes 1)\theta_1(w_{h_n})(\alpha_{h_n}(y) \otimes 1))\|_\varphi \to 0 \quad \text{for all } \ x, y \in \widetilde{M}_0 \ . \tag{A.7}$$

Denote by $W \subset \widetilde{P}$ the set of "reduced words" in $\widetilde{P} = P * L\mathbb{Z}$, i.e. products of factors alternatingly belonging to $P_0 \ominus \mathbb{C}1$ and $\{u_n \mid n \in \mathbb{Z} \setminus \{0\}\}$. We also consider $1 \in W$ as the "empty word". We have $W = W_1 \sqcup W_2$ where the elements $w \in W_1$ belong to $\theta_1(P)P$ and the elements $w \in W_2$ are orthogonal to $\theta_1(P)P$. We finally denote, for $\mathcal{F} \subset I$, by $W^{\mathcal{F}} \subset \widetilde{M}$ the set of elementary tensors with tensor factors in positions $i \in \mathcal{F}$ belonging to W. By density, it suffices to prove (A.7) for all finite subsets $\mathcal{F} \subset I$ and all $x, y \in W^{\mathcal{F}}$. Since $h_n \cdot i \to \infty$ for all $i \in I$, we have that $\mathcal{F} \cap h_n \cdot \mathcal{F} = \emptyset$ for all n large enough. In what follows, we may thus assume that $\mathcal{F} \cap h_n \cdot \mathcal{F} = \emptyset$. If at least one of the factors of $x \in W^{\mathcal{F}}$ belongs to W_2, using $\mathcal{F} \cap h_n \cdot \mathcal{F} = \emptyset$, one checks that

$$E_{M \overline{\otimes} Q}((x^\star \otimes 1)\theta_1(w)(\alpha_{h_n}(y) \otimes 1)) = 0 \quad \text{for all } \ w \in M \mathbin{\overline{\otimes}} Q \ .$$

The same holds if one of the factors of $y \in W^{\mathcal{F}}$ belongs to W_2. So to conclude the proof of (A.7), we may assume that $x = \theta_1(a)b$ and $y = \theta_1(c)d$ with $a, b, c, d \in M_0$. But then,

$$\begin{aligned} & E_{M \overline{\otimes} Q}((x^\star \otimes 1)\theta_1(w_{h_n})(\alpha_{h_n}(y) \otimes 1)) \\ & \qquad = (b^\star \otimes 1)\,E_{M \overline{\otimes} Q}(\theta_1((a^\star \otimes 1)\,w_{h_n}\,(\alpha_{h_n}(c) \otimes 1)))\,(\alpha_{h_n}(d) \otimes 1) \ . \end{aligned}$$

Since

$$E_{M\overline{\otimes}Q}(\theta_1((a^\star \otimes 1)\, w_{h_n}\, (\alpha_{h_n}(c) \otimes 1)))$$
$$= 1 \otimes (\varphi \otimes \mathrm{id})((a^\star \otimes 1)\, w_{h_n}\, (\alpha_{h_n}(c) \otimes 1)) \;,$$

we conclude that (A.7) follows from the assumption in (A.6).

By density, (A.7) implies that

$$\|E_{M\overline{\otimes}Q}(x^\star \theta_1(w_{h_n})\alpha_{h_n}(y))\|_\varphi \to 0 \quad \text{for all } \; x, y \in (\widetilde{M \,\overline{\otimes}\, Q})_0 \;. \tag{A.8}$$

Taking $x = y = V$ and using (A.5), it follows that $\|p_0\|_2 = \|p_0 w_{h_n}\|_2 \to 0$. This is absurd and therefore, there is no sequence $h_n \in \Sigma$ satisfying (A.6).

Step 5. We denote by Δ_φ the modular operator of φ on $L^2(M)$ and by $\widehat{\varphi}$ the normal semifinite faithful weight $\mathrm{Tr}(\Delta_\varphi \cdot) \otimes \varphi$ on $B(L^2(M)) \,\overline{\otimes}\, Q$. Denote $\mathcal{N} := (B(L^2(M)) \,\overline{\otimes}\, Q)_{\widehat{\varphi}}$ and note that $\widehat{\varphi}$ is a semifinite trace on $\mathcal{N}$. We denote by $a_h \in \mathcal{U}(L^2(M))$ the automorphism α_h viewed as a unitary operator on $L^2(M)$. Using the formula $\mathrm{Ad}\, a_g \otimes \alpha_g$, the automorphism α_g of $M \,\overline{\otimes}\, Q$ is naturally extended to $B(L^2(M)) \,\overline{\otimes}\, Q$. We still denote this extension by α_g. We claim that there exists a nonzero projection $T \in p\mathcal{N}p$ satisfying

$$\widehat{\varphi}(T) < \infty \quad \text{and} \quad T = w_h\, \alpha_h(T)\, w_h^\star \quad \text{for all } \; h \in \Sigma \;. \tag{A.9}$$

Consider $\Sigma \curvearrowright \ell^2(I)$ given by $\Sigma \curvearrowright I$. Since there is no sequence $h_n \in \Sigma$ satisfying (A.6), there exists $\delta > 0$, a finite subset $\mathcal{F} \subset I$ and finitely many elements $x_1, \ldots, x_m \in M$ that are eigenvectors for the modular group of φ with eigenvalues $\lambda_1, \ldots, \lambda_m$, such that

$$\langle h \cdot 1_\mathcal{F}, 1_\mathcal{F} \rangle + \sum_{i,j=1}^m \lambda_i \, \|(\varphi \otimes \mathrm{id})((x_j^\star \otimes 1) w_h (\alpha_h(x_i) \otimes 1))\|_\varphi^2 \geq \delta \;\text{ for all }\; h \in \Sigma \;. \tag{A.10}$$

Whenever $x \in M$ is an eigenvector for the modular group of φ, we consider the rank one operator T_x in $B(L^2(M))$ given by $T_x(y) = x\varphi(x^\star y)$. Note that $T_x \otimes 1 \in \mathcal{N}$. Define, in the Hilbert space $\mathcal{K} = \ell^2(I) \oplus L^2(p\mathcal{N}p, \widehat{\varphi})$, the vector

$$T_0 = 1_\mathcal{F} \oplus p\Big(\sum_{i=1}^m T_{x_i} \otimes 1\Big)p \;.$$

The formula $h \cdot (\xi, S) = (h \cdot \xi, w_h \alpha_h(S) w_h^\star)$ defines a unitary representation of Σ on $\mathcal{K}$. Formula (A.10) says that

$$\langle h \cdot T_0, T_0 \rangle \geq \delta \quad \text{for all } \; h \in \Sigma \;.$$

Therefore, the unitary representation $\Sigma \curvearrowright \mathcal{K}$ has a nonzero invariant vector. Since $\Sigma \curvearrowright \ell^2(I)$ has no nonzero invariant vectors, the representation $\Sigma \curvearrowright L^2(p\mathcal{N}p, \widehat{\varphi})$ has a nonzero invariant vector. Using functional calculus, it follows that (A.9) holds.

Step 6. There exists a nonzero projection $p_0 \in (M \mathbin{\overline{\otimes}} Q)_\varphi$ with $p_0 \leq p$, a Hilbert space H, a projection $q \in Q_\varphi \mathbin{\overline{\otimes}} B(H)$ with $(\varphi \otimes \mathrm{Tr})(q) < \infty$, a positive number $\lambda > 0$, an element $V \in (M \mathbin{\overline{\otimes}} Q)_{\varphi,\lambda} \otimes \overline{H}$ with $VV^\star = p_0$, $V^\star V = 1 \otimes q$ and a generalized 1-cocycle $(W_h)_{h\in\Sigma}$ for the action $(\alpha_h)_{h\in\Sigma}$ on $Q_\varphi \mathbin{\overline{\otimes}} B(H)$ with support projection q, such that

$$w_h\, \alpha_h(V) = V(1 \otimes W_h) \quad \text{for all } \ h \in \Sigma\ . \tag{A.11}$$

Denote by P_λ the orthogonal projection of $L^2(M \mathbin{\overline{\otimes}} Q)$ onto $L^2((M \mathbin{\overline{\otimes}} Q)_{\varphi,\lambda})$. Take a nonzero projection $T \in p\mathcal{N}p$ satisfying (A.9). Note that $\mathcal{N}$ commutes with the projections P_λ. Choose $\lambda > 0$ such that TP_λ is nonzero. By construction, TP_λ is the orthogonal projection onto a closed subspace $\mathcal{E} \subset L^2(p(M \mathbin{\overline{\otimes}} Q)_{\varphi,\lambda})$ that is a right $(1 \otimes Q_\varphi)$-module and that is globally invariant under $\xi \mapsto w_h\, \alpha_h(\xi)$ for all $h \in \Sigma$.

We claim that $\mathcal{E}$ has finite $(1 \otimes Q_\varphi)$-dimension. Denote by $\mathcal{S}$ the commutant of the right $(1 \otimes Q_\varphi)$-action on $L^2((M \mathbin{\overline{\otimes}} Q)_{\varphi,\lambda})$. Note that $\mathcal{N}P_\lambda \subset \mathcal{S}$. The normal faithful tracial state φ on Q_φ induces the normal semifinite faithful trace $\widehat{\tau}$ on $\mathcal{S}$. To prove the claim, we must show that $\widehat{\tau}(TP_\lambda) < \infty$. Denote by $E_\lambda \in \mathcal{Z}(\mathcal{N})$ the smallest projection in $\mathcal{N}$ that dominates P_λ. Applying Lemma A.4 to $1 \otimes Q \subset M \mathbin{\overline{\otimes}} Q$, we get that

$$\widehat{\tau}(SP_\lambda) = \frac{1}{\lambda}\widehat{\varphi}(SE_\lambda) \quad \text{for all } \ S \in \mathcal{N}^+\ . \tag{A.12}$$

In particular, the claim that $\widehat{\tau}(TP_\lambda) < \infty$ follows from the property that $\widehat{\varphi}(T) < \infty$.

Choose a Hilbert space H, a nonzero projection $q \in Q_\varphi \mathbin{\overline{\otimes}} B(H)$ with $(\varphi \otimes \mathrm{Tr})(q) < \infty$ and a Q_φ-linear unitary operator $\Psi : q(L^2(Q_\varphi) \otimes H) \to \mathcal{E}$. Fix $h \in \Sigma$. The unitary operator

$$\alpha_h(q)(L^2(Q_\varphi) \otimes H) \to q(L^2(Q_\varphi) \otimes H) : \xi \mapsto \Psi^{-1}\big(w_h(\alpha_h \circ \Psi \circ \alpha_h^{-1})(\xi)\big)$$

is right Q_φ-linear. So it corresponds to left multiplication with $W_h \in Q_\varphi \mathbin{\overline{\otimes}} B(H)$ satisfying $W_h W_h^\star = q$ and $W_h^\star W_h = \alpha_h(q)$. By construction, (W_h) is a generalized 1-cocycle for the action $(\alpha_h)_{h\in\Sigma}$ with support projection q.

The operator $\Theta \in B(H, pL^2((M \mathbin{\overline{\otimes}} Q)_{\varphi,\lambda}))$ given by $\Theta(\xi) = \Psi(q(1 \otimes \xi))$ satisfies

$$\mathrm{Tr}(\Theta^\star\Theta) = (\varphi \otimes \mathrm{Tr})(q) < \infty\ .$$

Therefore, Θ can be viewed as a nonzero vector $V \in p(L^2((M \overline{\otimes} Q)_{\varphi,\lambda}) \otimes \overline{H})(1 \otimes q)$ satisfying

$$w_h \, \alpha_h(V) = V(1 \otimes W_h) \quad \text{for all } \; h \in \Sigma \; . \tag{A.13}$$

The left support of V is a projection $p_0 \in (M \mathbin{\overline{\otimes}} Q)_\varphi$ satisfying $p_0 \leq p$. The right support of V is a projection $q_1 \in (M \mathbin{\overline{\otimes}} Q)_\varphi \mathbin{\overline{\otimes}} B(H)$ satisfying $q_1 \leq 1 \otimes q$. By (A.13), we also have that $\alpha_h(q_1) = (1 \otimes W_h^\star) q_1 (1 \otimes W_h)$ for all $h \in \Sigma$. Lemma A.2.2 implies that $q_1 = 1 \otimes q_2$ for some projection $q_2 \in Q_\varphi \mathbin{\overline{\otimes}} B(H)$ satisfying $q_2 \leq q$. But then $\Psi(\xi) = 0$ for all ξ in the image of $q - q_2$. Since Ψ is unitary, we conclude that $q_2 = q$. Taking the polar part of V, we have found the partial isometry V satisfying (A.11).

Step 7. We upgrade step 6 from Σ to the entire group Λ. More precisely, we prove that there exists a nonzero projection $p_0 \in (M \mathbin{\overline{\otimes}} Q)_\varphi$ with $p_0 \leq p$, a Hilbert space H, a projection $q \in Q_\varphi \mathbin{\overline{\otimes}} B(H)$ with $(\varphi \otimes \mathrm{Tr})(q) < \infty$, a positive number $\lambda > 0$, an element $V \in (M \mathbin{\overline{\otimes}} Q)_{\varphi,\lambda} \otimes \overline{H}$ with $VV^\star = p_0$, $V^\star V = 1 \otimes q$ and a generalized 1-cocycle $(W_g)_{g \in \Lambda}$ for the action $(\alpha_g)_{g \in \Lambda}$ on $Q_\varphi \mathbin{\overline{\otimes}} B(H)$ with support projection q, such that

$$w_g \, \alpha_g(V) = V(1 \otimes W_g) \quad \text{for all } \; g \in \Lambda \; . \tag{A.14}$$

Denote by β_g the action of Λ on $p(M \mathbin{\overline{\otimes}} Q)_\varphi p$ given by $\beta_g = (\mathrm{Ad}\, w_g) \circ \alpha_g$. Fix the positive number $\lambda > 0$ that appears in the statement of step 6. Denote by $\mathcal{P}$ the set of projections $p_0 \in (M \mathbin{\overline{\otimes}} Q)_\varphi$ with $p_0 \leq p$ such that there exist H, q and W_h satisfying the conclusion of step 6. By (A.11), all $p_0 \in \mathcal{P}$ satisfy $\beta_h(p_0) = p_0$ for all $h \in \Sigma$. We make the following four observations about $\mathcal{P}$.

Take $g \in \Gamma$. Since Γ and Σ commute, we can apply $x \mapsto w_g \alpha_g(x)$ to (A.11) and conclude that $\beta_g(p_0) \in \mathcal{P}$ for all $p_0 \in \mathcal{P}$.

We next prove that if $p_0 \in \mathcal{P}$ and $p_1 \leq p_0$ is a projection satisfying $\beta_h(p_1) = p_1$ for all $h \in \Sigma$, then also $p_1 \in \mathcal{P}$. This follows by multiplying (A.11) on the left by p_1 and using Lemma A.2.2 to conclude that the right support of $p_1 V$ belongs to $1 \mathbin{\overline{\otimes}} Q_\varphi \mathbin{\overline{\otimes}} B(H)$ and can be used to cut down W_h.

We next prove that if $p_1, p_2 \in \mathcal{P}$, then also $p_0 = p_1 \vee p_2$ belongs to $\mathcal{P}$. Taking the direct sum of the V, H, q and W_h that come with p_1 and p_2, we find an element $V \in (M \mathbin{\overline{\otimes}} Q)_{\varphi,\lambda} \mathbin{\overline{\otimes}} \overline{H}$ and a generalized 1-cocycle W_h with support q such that (A.11) holds and such that the left support of V equals p_0. As in the previous paragraph, the right support of V belongs to $1 \mathbin{\overline{\otimes}} Q_\varphi \mathbin{\overline{\otimes}} B(H)$. Cutting down with this projection and replacing V by its polar part, we find that $p_0 \in \mathcal{P}$.

We finally make the obvious observation that $\mathcal{P}$ is closed under taking a direct sum of projections that are orthogonal.

By step 6, the set $\mathcal{P}$ is nonempty. Combining the four observations above, we find a nonzero projection $p_0 \in \mathcal{P}$ that satisfies $\beta_g(p_0) = p_0$ for all $g \in \Gamma$. Take the corresponding V, H, q and W_h. To prove that (A.14) holds, it suffices to prove that $V^\star w_g \alpha_g(V)$ belongs to $1 \overline{\otimes} Q_\varphi \overline{\otimes} B(H)$ for all $g \in \Lambda$. Since we already know this when $g \in \Sigma$, it remains to consider $g \in \Gamma$. Put $X = V^\star w_g \alpha_g(V)$. Since Γ and Σ commute, and using (A.11), we find that

$$X = \kappa(g,h)\,(1 \otimes W_h)\,\alpha_h(X)\,(1 \otimes \alpha_g(W_h)^\star) \quad \text{for all } \ h \in \Sigma\ .$$

By Lemma A.2.2, it follows that $X \in 1 \overline{\otimes} Q_\varphi \overline{\otimes} B(H)$. This concludes the proof of step 7.

End of the proof. Having found p_0 as in step 7, we have $p_0 w_g = w_g \alpha_g(p_0)$ for all $g \in \Lambda$ and then $((p - p_0)w_g)_{g\in\Lambda}$ is a new generalized 1-cocycle with support projection $p - p_0$. We can apply the reasoning above to $(p - p_0)w_g$. Therefore, the theorem follows by a maximality argument. ☐

Corollary A.3. *Let $\alpha : \Lambda \curvearrowright (M, \varphi) = (P, \phi)^I$ be a generalized Bernoulli action satisfying the same assumptions as in Theorem A.1. Let $w_g \in M_\varphi$ be a generalized 1-cocycle with support projection $p \in M_\varphi$ and scalar 2-cocycle Ω.*

If the action $\operatorname{Ad} w_g \circ \alpha_g$ of Λ on $pM_\varphi p$ has a trivial fixed point algebra, then there exists an integer $n \in \mathbb{N}$, an irreducible Ω-representation $\pi : \Lambda \to \mathcal{U}(n)$ and, writing $\lambda = \varphi(p)/n$, an element $V \in M_{\varphi,\lambda} \otimes \overline{\mathbb{C}^n}$ satisfying $VV^\star = p$, $V^\star V = 1 \otimes 1$ and

$$w_g = V(1 \otimes \pi(g))\alpha_g(V^\star) \quad \textit{for all } \ g \in \Lambda\ .$$

If the action $\operatorname{Ad} w_g \circ \alpha_g$ of Λ on $pM_\varphi p$ even has no nontrivial finite-dimensional globally invariant subspaces, then $n = 1$ and $\pi(g) \in \mathbb{T}$ satisfies $\pi(g)\pi(g') = \Omega(g, g')\pi(gg')$ for all $g, g' \in \Lambda$.

Proof. By Theorem A.1, we find projections $p_k \in M_\varphi$ such that $p = \sum_k p_k$ and $p_k = w_g \alpha_g(p_k) w_g^\star$, and we find Hilbert spaces H_k, positive numbers $\lambda_k > 0$, numbers $n_k \in \mathbb{N}_0$ and elements $v_k \in M_{\varphi,\lambda_k} \overline{\otimes}\, \overline{\mathbb{C}^{n_k}}$ such that

$$v_k v_k^\star = p_k \text{ and } v_k^\star v_k = 1 \otimes 1, \quad v_k^\star w_g \alpha_g(v_k) = 1 \otimes W_{k,g}$$

for $(W_{k,g})_{g\in\Lambda} \in M_{n_k}(\mathbb{C})$ unitaries satisfying

$$W_{k,g} W_{k,g'} = \Omega(g, g') W_{k,gh} \quad \text{for } g, g' \in \Lambda.$$

Assume that the action $\operatorname{Ad} w_g \circ \alpha_g$ on $pM_\varphi p$ has trivial fixed point algebra. Since all the p_k's are invariant under $\operatorname{Ad} w_g \circ \alpha_g$, necessarily $p = p_1$. Because

$v_1 \in M_{\varphi,\lambda_k} \otimes \overline{\mathbb{C}^{n_1}}$, we have $\varphi(p) = \varphi(v_1 v_1^\star) = \lambda_1 (\varphi \otimes \mathrm{Tr})(v_1^\star v_1) = \lambda_1 n_1$. Putting $V = v_1, \lambda = \lambda_1, n = n_1$ and $\pi(g) = W_{1,g}$, we arrive at the conclusions of the corollary.

It is easy to see that $V(1 \otimes M_n(\mathbb{C}))V^\star$ is a finite-dimensional subspace of $pM_\varphi p$, globally invariant under $\mathrm{Ad}\, w_g \circ \alpha_g$. If no such subspace exists, then it follows that $n = 1$, and $\pi(g) \in \mathbb{T}$. □

In the proof of Theorem A.1, we used the following general lemma. Assume that (N, φ) is a von Neumann algebra with a faithful normal almost periodic state, that $Q \subset N$ is a von Neumann subalgebra and that $E : N \to Q$ is a state-preserving conditional expectation. Note that $\sigma_t^\varphi(Q) = Q$ and that the restriction of σ_t^φ to Q equals the modular automorphism group of Q.

Denote by J and Δ the modular conjugation and modular operator on $L^2(N)$. Denote by $e : L^2(N) \to L^2(Q)$ the orthogonal projection of $L^2(N)$ onto $L^2(Q)$. The von Neumann algebra $\langle N, e\rangle$ acting on $L^2(N)$ generated by N and e coincides with the commutant $JQ'J$ and is called the basic construction. It comes with a canonical normal semifinite faithful weight $\widehat{\varphi}$ characterized by the following two properties: $\sigma_t^{\widehat{\varphi}} = \mathrm{Ad}\, \Delta^{it}$ and $\widehat{\varphi}(xey) = \varphi(xy)$ for all $x, y \in N$.

For every $\lambda > 0$, define the projection $E_\lambda \in \langle N, e\rangle$ as the join of the support projections of all $xex^\star$, $x \in N_{\varphi,\lambda}$. Also, denote by P_λ the orthogonal projection of $L^2(N)$ onto $L^2(N_{\varphi,\lambda})$, and note that $P_\lambda \leq E_\lambda$. Define $\mathcal{S}_\lambda$ as the commutant of the right action of Q_φ on $L^2(N_{\varphi,\lambda})$. The restriction of φ to Q_φ is a normal faithful trace τ on Q_φ. This induces a normal semifinite faithful trace $\widehat{\tau}$ on $\mathcal{S}_\lambda$ characterized by the formula $\widehat{\tau}(VW^\star) = \tau(W^\star V)$ whenever $V, W : L^2(Q_\varphi) \to L^2(N_{\varphi,\lambda})$ are bounded, right Q_φ-linear operators.

Lemma A.4. *We have* $\langle N, e\rangle_{\widehat{\varphi}} P_\lambda = \mathcal{S}_\lambda$ *and* $\widehat{\tau}(TP_\lambda) = \frac{1}{\lambda}\widehat{\varphi}(TE_\lambda)$ *for all* $T \in \langle N, e\rangle_{\widehat{\varphi}}^+$.

Proof. Note that $\langle N, e\rangle_{\widehat{\varphi}}$ is generated by $\{aeb^\star \mid \mu > 0\ ,\ a, b \in N_{\varphi,\mu}\}$. Define $\mathcal{T}_\lambda$ as the von Neumann algebra generated by $\{xey^\star \mid x, y \in N_{\varphi,\lambda}\}$. By definition, E_λ is the unit of $\mathcal{T}_\lambda$. When $\mu > 0$, $a, b \in N_{\varphi,\mu}$ and $x, y \in N_{\varphi,\lambda}$, we have

$$aeb^\star\, xey^\star = aE(b^\star x)ey^\star\ .$$

It follows that E_λ is a central projection in $\langle N, e\rangle_{\widehat{\varphi}}$ and $\langle N, e\rangle_{\widehat{\varphi}}\, E_\lambda = \mathcal{T}_\lambda$. For $x \in N_{\varphi,\lambda}$, define $V_x : L^2(Q_\varphi) \to L^2(N_{\varphi,\lambda})$ by $V_x(a) = xa$. For all $x, y \in N_{\varphi,\lambda}$, we have $xey^\star\, P_\lambda = V_x V_y^\star$. It follows that $\mathcal{T}_\lambda\, P_\lambda = \mathcal{S}_\lambda$ and

$$\widehat{\tau}(xey^\star P_\lambda) = \widehat{\tau}(V_x V_y^\star) = \tau(V_y^\star V_x) = \varphi(y^\star x) = \frac{1}{\lambda}\, \varphi(xy^\star)\ .$$

Since $\widehat{\varphi}(xey^\star) = \varphi(xy^\star)$, the lemma is proved. $\square$

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- Classification of type III Bernoulli crossed products. With Stefaan Vaes. *Adv. Math.*, 281:296–332, 2015.
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